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ORIGINAL ARTICLE

A fragile multi-CPR game

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Abstract

A Fragile CPR Game is an instance of a resource sharing game where a common-pool resource, which is prone to failure due to overuse, is shared among several players. Each player has a fixed initial endowment and is faced with the task of investing in the common-pool resource without forcing it to fail. The return from the common-pool resource is subject to uncertainty and is perceived by the players in a prospect-theoretic manner. It has already been shown in the existing literature that, under some mild assumptions, a Fragile CPR Game admits a unique Nash equilibrium. In this article we investigate an extended version of a Fragile CPR Game, in which players are allowed to share multiple common-pool resources that are also prone to failure due to overuse. We refer to this game as a Fragile multi-CPR Game. Our main result states that, under some mild assumptions, a Fragile multi-CPR Game admits a Generalized Nash

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equilibrium. Moreover, we show that, when there are more players than common-pool resources, the set consisting of all Generalized Nash equilibria of a Fragile multi-CPR Game is of Lebesgue measure zero.

Keywords CPR games · prospect theory · Generalized Nash equilibrium

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1 Prologue, related work and main results

In this article we shall be concerned with a *resource sharing game*. Such games model instances in which a common-pool resource (henceforth CPR), which is prone to failure due to overuse, is shared among several users who are addressing the problem of choosing how much to exploit from/invest in the CPR without forcing it to fail. Resource sharing games arise in a variety of problems ranging from economics to computer science. Examples of CPRs include arable lands, forests, fisheries, groundwater basins, spectrum and computing resources, the atmosphere, among many others. Such CPRs are, on the one hand, usually regenerative but, on the other hand, subject to failure (i.e., such CPRs are fragile) when several agents exploit the resource in an unsustainable manner. Each agent exploits/invests in the CPR in order to obtain an individual benefit. However, it has been observed that actions which are individually rational (e.g. Nash equilibria) may result in outcomes that are collectively irrational, thus giving rise to a particular social dilemma known as "*the tragedy of the commons*" [see Hardin (1968)]. It is thus of interest to investigate equilibrium points of resource sharing games, in order to better understand situations where such a social dilemma arises. This is a topic that has drawn considerable attention, both from a theoretical and a practical perspective. We refer the reader to Aflaki (2013), Budescu et al. (1995), Hota et al. (2016), Hota and Sundaram (2020), Keser and Gardner (1999), Ostrom (1990), Ostrom et al. (1994), Vamvakas et al. (2019a), Vamvakas et al. (2019b) and Walker and Gardner (1992) for applications, variations, and for further references on resource sharing games.

One particular motivating example for considering multi-CPR games arises in the study of next generation wireless networks, including the emerging 5G/6G wireless systems. In this context, a typical wireless cellular network divides a particular internet-service area into small geographical sub-areas, which are referred to as cells. Those cells serve different wireless devices (which are referred to as users) that aim to transfer and receive data to/from a Base Station using wireless transmission power and consuming some part of the available system bandwidth (i.e., spectrum bands). The available bandwidth may be split in different parts (for instance either licensed bands and/or unlicensed bands), with each band having different characteristics and operation benefits, and offered to the various users under different operational principles. In that sense users may exploit simultaneously different parts of the available spectrum bands, while at the same time they may use multiple access technologies in order to gain access to these resources. Examples of well-known, and extensively deployed such technologies, include NOMA (Ding et al. 2017), OFMDA (Tsiropoulou et al. 2016), CDMA and OMA (Vamvakas et al. 2018), among many others. The reader could think of any of the aforementioned alternatives in accessing the system resources as a "supplier" which has an available amount of limited resources that is supplied to, and is shared among, the users according to their demand, and which is therefore subject to failure due to excessive demand by the users. In other words, each “supplier" may be seen as a CPR that is subject to “the tragedy of the commons", and the users are thus faced with the problem of choosing how much to exploit from each resource without forcing it to fail. It is important to note that the performance of each “supplier" is subject to uncertainty as well as that it is independent of the performances of the other “suppliers". A common approach to model 5G systems is based on resource sharing games: each access technology (or each spectrum band) is a CPR and each user is a player. However, most results in the literature appear to focus on games in which players exploit a single CPR, and it is natural to ask what happens when multiple CPRs are taken into consideration simultaneously. It is rather intuitive that players who are allowed to exploit more than one options should achieve higher utilities as well as lower chances of forcing the corresponding shared resource to fail. We will see in an example below that multi-CPR games do possess such properties.

In this article we investigate a resource sharing game in which the players are allowed to invest in/exploit from several CPRs, whose performances are *mutually independent*. To the best of our knowledge, our paper appears to be among the first to consider resource sharing games played on more than one CPR, and one of the aims of this work is to provide a solid mathematical background to initiate such a shift. Although this generalization seems important for the applicability of this area to real world scenarios, it seemingly complicates the analysis, pinpointing to the necessity of using mathematical tools that are substantially different from the ones in the current bibliography.

We shall be interested in a multi-CPR version of a particular resource sharing game, which is referred to as a *Fragile CPR Game*. It is introduced in Hota et al. (2016) and may be seen as a prospect-theoretic version of a well-established game-theoretic model for resource sharing, which is referred to as the *Standard CPR Game* [see Ostrom et al. (1994, p. 109)]. The Fragile CPR Game is played by several players, each of whom has a fixed initial endowment and must decide how much to invest in the CPR. The performance of the CPR is subject to uncertainty; that is, there is a probability that the CPR will fail, and this probability depends on the total investment in the CPR. If the CPR fails, then the players lose their investment. If the CPR does not fail, then there is a rate of return, which also depends on the total investment in the CPR, and is perceived by the players in a prospect-theoretic manner. The failure probability is assumed to be an increasing and convex function of the total investment, and the rate of return is assumed to be a monotone function of the total investment. This includes linearly increasing failure probability functions, which have been considered in the study of resource dilemma problems [see Budescu et al. (1995)], as well as decreasing rate functions, which have been considered both in theoretical studies as well in applications of resource sharing games (Nisan et al. 2007; Ostrom et al. 1994; Vamvakas et al. 2019a, b). Besides being interesting from a theoretical perspective, the Fragile CPR Game has been proven to be useful in several real world problems including the design of tax mechanisms [see Hota and Sundaram (2020))] the quality of experience in social systems [see Thanou et al. (Feb. 2019)], the management and control of spectrum fragility in 5G wireless networks [see Vamvakas et al. (2019a, b)], among others.

The main result in Hota et al. (2016) states that a Fragile CPR Game admits a unique Nash equilibrium. In this article we focus on an extended version of a Fragile CPR Game. We refer to the corresponding game as a *Fragile multi-CPR Game* and investigate its *Generalized Nash equilibria*. Our main result states that the set consisting of all Generalized Nash equilibria of a Fragile multi-CPR Game is non-empty and, when there are more players than CPRs, "small" in a measure-theoretic sense. In the next subsection we introduce the Fragile CPR Game and state the main result from Hota et al. (2016). We then proceed, in Sect. 1.2, with defining the Fragile multi-CPR Game, which is the main target of this work, and stating our main results.

1.1 Fragile CPR game

Throughout the text, given a positive integer n, we denote by $[n]$ the set $\{1, \dots, n\}$. In this section we define the *Fragile CPR Game*. It is introduced in Hota et al. (2016), and is played by n players, who are assumed to be indexed by the set $[n]$. It is also assumed that there is a single CPR, and each player has to decide how much to invest in the CPR. Each player has an available endowment, which, without loss of generality, is assumed to be equal to 1. Every player, say $i \in [n]$, invests an amount $x_i \in [0, 1]$ in the CPR. The total investment of all players in the CPR is denoted $\mathbf{x}_T = \sum_{i\in[n]} x_i$. The performance of the CPR is subject to uncertainty, that is there is a probability $p(\mathbf{x}_T)$ that the CPR will fail, and this probability depends on the total investment of the players in the CPR. In case the CPR fails, the players lose their investment in the CPR. In case the CPR does not fail, then there is a *rate of return* from the CPR which depends on the total investment of all players, and is denoted by $\mathcal{R}(\mathbf{x}_T)$. The rate of return is assumed to satisfy $\mathcal{R}(\mathbf{x}_T) > 1$, for all $\mathbf{x}_T \geq 0$.

In other words, player $i \in [n]$ gains $x_i \cdot \mathcal{R}(\mathbf{x}_T) - x_i$ with probability $1 - p(\mathbf{x}_T)$, and gains $-x_i$ with probability $p(\mathbf{x}_T)$. The situation is modelled through a prospect-theoretic perspective, in the spirit of Kahneman and Tversky (1979). More precisely, let $x^{(i)} = \sum_{j\in[n]\setminus\{i\}} x_j$; hence it holds $x_i + x^{(i)} = \mathbf{x}_T$. Then the utility of player $i \in [n]$ is given by the following utility function:

$$\mathcal{V}_i(x_i, x^{(i)}) = \begin{cases} (x_i \cdot (\mathcal{R}(\mathbf{x}_T) - 1))^{a_i}, & \text{with probability } 1 - p(\mathbf{x}_T), \\ -k_i x_i^{a_i}, & \text{with probability } p(\mathbf{x}_T). \end{cases} \quad (1)$$

Observe that, despite the fact that the rate of return $\mathcal{R}(\cdot)$ is assumed to satisfy $\mathcal{R}(0) > 1$, the utility of a player who invests zero in the CPR is equal to zero. The parameters k_i and a_i are fixed and player-specific. Let us note that the parameter k_i may be thought of as capturing the "behaviour" of each player. More precisely, when $k_i > 1$ then a player weighs losses more than gains, a behaviour which is referred to as "loss averse". On the other hand, when $k_i \in [0, 1]$ then a player weighs gains more than losses, a behaviour which is referred to as "gain seeking". Capturing behaviours of this type among players constitutes a central aspect of prospect theory [see, for example, Wakker (2010)]. Notice that when $k_i = 1$ and $a_i = 1$ then player $i \in [n]$ is *risk neutral*.

Each player of the Fragile CPR game is an expected utility maximizer, and therefore chooses $x_i \in [0, 1]$ that maximizes the expectation of $\mathcal{V}(x_i, x^{(i)})$, i.e, that maximizes the *utility* of player $i \in [n]$ which is given by

$$\mathbb{E}\left(\mathcal{V}_i(x_i, x^{(i)})\right) = x_i^{a_i} \cdot \mathcal{F}_i(\mathbf{x}_T)\,,$$

where

$$\mathcal{F}_i(\mathbf{x}_T) = (\mathcal{R}(\mathbf{x}_T) - 1)^{a_i} \cdot (1 - p(\mathbf{x}_T)) - k_i \cdot p(\mathbf{x}_T) \tag{2}$$

is the *effective rate of return* to payer $i \in [n]$.

The main result in Hota et al. (2016) establishes, among other things, the existence of a unique Nash equilibrium for the Fragile CPR game, provided the following hold true.

Assumption 1 Consider a Fragile CPR game that satisfies the following properties.

1. The function $p(\cdot)$ is twice continuously differentiable, and satisfies $p(0) = 0$ and $p(\mathbf{x}_T) = 1$, whenever $\mathbf{x}_T \geq 1$.
2. $a_i \in (0, 1]$ and $k_i > 0$, for all $i \in [n]$.
3. For all $i \in [n]$ and all $\mathbf{x}_T \in (0, 1)$ it holds $\frac{\partial}{\partial \mathbf{x}_T}\mathcal{F}_i(\mathbf{x}_T), \frac{\partial^2}{\partial \mathbf{x}_T^2}\mathcal{F}_i(\mathbf{x}_T) < 0$, where $\mathcal{F}_i$ is given by (2).

In other words, the first condition in Assumption 1 states that the CPR fails for sure, when the investment is "high", thus rendering the Fragile CPR Game to be subject to the “tragedy of the commons". Let us remark that the particular choice of the total investment of the players, $\mathbf{x}_T$, is decisive, since it may cause the CPR to either be in a *secure state* (i.e., a state for which $p(\mathbf{x}_T)$ is small), or a *fragile state* (i.e., a state for which $p(\mathbf{x}_T)$ is large). The third condition states that the effective rate of return of all players is a strictly decreasing and concave function. An example of an effective rate of return $\mathcal{F}_i$ satisfying the conditions of Assumption 1 is obtained by choosing $a_i < 1/2$, $p(\mathbf{x}_T) = \mathbf{x}_T^2$, and $\mathcal{R}(\mathbf{x}_T) = 2 - e^{\mathbf{x}_T - 1}$, as can be easily verified.

Before proceeding with the main result from Hota et al. (2016), let us recall here the notion of Nash equilibrium, adjusted to the setting of the Fragile CPR Game.

Definition 1 (Nash Equilibrium) A *Nash equilibrium* for a Fragile CPR Game is a strategy profile $(x_1, \ldots, x_n) \in [0, 1]^n$ such that for all $i \in [n]$ it holds:

$$\mathbb{E}\left(\mathcal{V}(x_i, x^{(i)})\right) \geq \mathbb{E}\left(\mathcal{V}(z_i, x^{(i)})\right)\,, \text{ for all } z_i \in [0, 1]\,.$$

In other words, $(x_1, \ldots, x_n) \in [0, 1]^n$ is a Nash equilibrium for a Fragile CPR Game if no player can increase her utility by unilaterally changing strategy. The main result in Hota et al. (2016) reads as follows.

Theorem 1 (Hota et al. 2016) *Consider a Fragile CPR Game that satisfies Assumption 1. Then the game admits a* unique *Nash equilibrium.*

We now proceed with defining the *Fragile multi-CPR Game*, whose equilibria are the main target of the present article.

1.2 Fragile multi-CPR game

In this subsection we define the *Fragile multi-CPR Game*. Throughout the text, the parameter n is fixed and will denote the number of players. Similarly, m is also fixed and will denote the number of CPRs. We begin with some extra piece of notation. If m is a positive integer, let C_m denote the set:

$$C_m = \left\{ (x_1, \ldots, x_m) \in [0,1]^m : \sum_{i\in[m]} x_i \le 1 \right\}. \tag{3}$$

Moreover, let $\mathcal{C}_n$ denote the Cartesian product $\prod_{i\in[n]} C_m$ and let $\mathcal{C}_{-i} = \prod_{[n]\setminus\{i\}} C_m$ denote the Cartesian product obtained from $\mathcal{C}_n$ by deleting its i-th component. Elements in $\mathcal{C}_{-i}$ are denoted by $\mathbf{x}_{-i}$, as is customary, and an element $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{C}_n$ is occasionally written $\mathbf{x} = (\mathbf{x}_i, \mathbf{x}_{-i})$, for $i \in [n]$, $\mathbf{x}_i \in C_m$ and $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$.

Suppose that there are n players, indexed by the set $[n]$, each having an *initial endowment* equal to 1. Assume further that there are m available CPRs, where $m \ge 1$ is an integer. Every player has to decide how much to invest in each CPR. More precisely, every player, say $i \in [n]$, chooses an element $\mathbf{x}_i = (x_{i1}, \ldots, x_{im}) \in C_m$ and invests x_{ij} in the j-th CPR. Given strategies $\mathbf{x}_i = (x_{i1}, \ldots, x_{im}) \in C_m, i \in [n]$, of the players and an integer $j \in [m]$, set

$$\mathbf{x}_T^{(j)} = \sum_{i\in[n]} x_{ij} \quad \text{and} \quad \mathbf{x}_T^{j|i} = \sum_{\ell\in[n]\setminus\{i\}} x_{\ell j}. \tag{4}$$

Hence it holds $\mathbf{x}_T^{(j)} = x_{ij} + \mathbf{x}_T^{j|i}$, for all $i \in [n]$. In other words, $\mathbf{x}_T^{(j)}$ equals the total investment of the players in the j-th CPR and $\mathbf{x}_T^{j|i}$ equals the total investment of all players except player i in the j-th CPR. As in the case of the Fragile CPR Game, we assume that the performance of each CPR is subject to uncertainty, and that each CPR has a corresponding rate of return, both depending on the total investment of the players in each CPR. More precisely, for $j \in [m]$, let $\mathcal{R}_j(\mathbf{x}_T^{(j)})$ denote the *return rate* of the j-th CPR and let $p_j(\mathbf{x}_T^{(j)})$ denote the *probability that the j-th CPR fails*. We assume that $\mathcal{R}_j(\mathbf{x}_T^{(j)}) > 1$ holds true, for all $\mathbf{x}_T^{(j)} \ge 0$.

The *utility* of player $i \in [n]$ from the j-th CPR is given, as in the case of the Fragile CPR game, via the following prospect-theoretic utility function:

$$\mathcal{V}_{ij}(x_{ij}, \mathbf{x}_T^{j|i}) = \begin{cases} (x_{ij} \cdot (\mathcal{R}_j(\mathbf{x}_T^{(j)}) - 1))^{a_i}, & \text{with probability } 1 - p_j(\mathbf{x}_T^{(j)}), \\ -k_i x_{ij}^{a_i}, & \text{with probability } p_j(\mathbf{x}_T^{(j)}). \end{cases} \tag{5}$$

Observe that when $x_{ij} = 0$, for some $i \in [n]$ and $j \in [m]$, then $\mathcal{V}_{ij}(x_{ij}, \mathbf{x}_T^{j|i}) = 0$, regardless of the performance of the CPR. We assume that the performance of each

CPR is *independent* of the performances of all remaining CPRs. Players in the Fragile multi-CPR Game are expected utility maximizers. If player $i \in [n]$ plays the vector $\mathbf{x}_i = (x_{i1}, \ldots, x_{im}) \in C_m$, and the rest of the players play $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$ then her expected utility from the j-th CPR is equal to

$$\mathcal{E}_{ij}(x_{ij}; \mathbf{x}_T^{j|i}) := \mathbb{E}\left(\mathcal{V}_{ij}(x_{ij}, \mathbf{x}_T^{j|i})\right) = x_{ij}^{a_i} \cdot \mathcal{F}_{ij}(\mathbf{x}_T^{(j)}), \tag{6}$$

where

$$\mathcal{F}_{ij}(\mathbf{x}_T^{(j)}) := (\mathcal{R}_j(\mathbf{x}_T^{(j)}) - 1)^{a_i}(1 - p_j(\mathbf{x}_T^{(j)})) - k_i p_j(\mathbf{x}_T^{(j)}) \tag{7}$$

is the *effective rate of return* to the i-th player from the j-th CPR. Notice that, since we assume that the performance of each CPR is independent of the performances of the remaining CPRs, $\mathcal{E}_{ij}$ depends only on the values of $x_{ij}, \mathbf{x}_T^{j|i}$ and does not depend on the values of $x_{ik}, \mathbf{x}_T^{k|i}$, for $k \neq j$. In other words, the (total) prospect-theoretic *utility* of player $i \in [n]$ in the Fragile multi-CPR Game is given by:

$$\mathcal{V}_i(\mathbf{x}_i; \mathbf{x}_{-i}) = \sum_{j \in [m]} \mathcal{E}_{ij}(x_{ij}; \mathbf{x}_T^{j|i}). \tag{8}$$

In this article we establish the existence of a Generalized Nash equilibrium for the Fragile multi-CPR game, provided the following holds true.

Assumption 2 Consider a Fragile multi-CPR Game that satisfies the following properties:

1. For every $j \in [m]$, the function $p_j(\cdot)$ is twice continuously differentiable, and satisfies $p_j(0) = 0$ and $p_j(\mathbf{x}_T^{(j)}) = 1$, whenever $\mathbf{x}_T^{(j)} \geq 1$.
2. It holds $a_i \in (0, 1]$ and $k_i > 0$, for all $i \in [n]$.
3. For all $i \in [n]$ and all $j \in [m]$ it holds $\frac{\partial}{\partial \mathbf{x}_T^{(j)}} \mathcal{F}_{ij}(\mathbf{x}_T^{(j)})$, $\frac{\partial^2}{\partial (\mathbf{x}_T^{(j)})^2} \mathcal{F}_{ij}(\mathbf{x}_T^{(j)}) < 0$, where $\mathcal{F}_{ij}$ is given by (7).

Notice that, similarly to the Fragile CPR Game, the first condition in Assumption 2 states that each CPR is subject to the "tragedy of the commons". The third condition states that the effective rate of return of every player from any CPR is a *strictly decreasing and concave* function. An example of an effective rate of return satisfying Assumption 2 is obtained by choosing, for $j \in [m]$, the return rate of the j-th CPR to be equal to $\mathcal{R}_j(\mathbf{x}_T^{(j)}) = c_j + 1$, where $c_j > 0$ is a constant, and the probability that the j-th CPR fails to be a strictly increasing and convex, on the interval [0, 1], function such that $p_j(\mathbf{x}_T^{(j)}) = 1$, when $\mathbf{x}_T^{(j)} \geq 1$.

Before stating our main result, let us proceed with recalling the notion of Generalized Nash equilibrium [see Facchinei and Kanzow (2007)].

Consider the, above-mentioned, Fragile multi-CPR Game, denoted G. Assume further that, for each player $i \in [n]$, there exists a correspondence $\vartheta_i : \mathcal{C}_{-i} \to 2^{C_m}$ mapping every element $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$ to a set $\vartheta_i(\mathbf{x}_{-i}) \subset C_m$. The set-valued correspondence ϑ_i is referred to as a *constraint policy* and may be thought of as determining the set of strategies that are feasible for player $i \in [n]$, given $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$. We refer to the tuple $(G, \{\vartheta_i\}_{i\in[n]})$ as the *Constrained Fragile multi-CPR Game* with constraint policies $\{\vartheta_i\}_{i\in[n]}$. Corresponding to a constrained game is the following notion of *Constrained Nash equilibrium* (or *Generalized Nash equilibrium*):

Definition 2 [GNE] A *Generalized Nash equilibrium* for a Constrained Fragile multi-CPR Game $(G, \{\vartheta_i\}_{i\in[n]})$ is a strategy profile $\mathbf{x}^* = (\mathbf{x}_1^*, \dots, \mathbf{x}_n^*) \in \mathcal{C}_n$ such that

1. For all $i \in [n]$, it holds $\mathbf{x}_i^* \in \vartheta_i(\mathbf{x}_{-i}^*)$, for all $i \in [n]$, and
2. For all $i \in [n]$, it holds $\mathcal{V}_i(\mathbf{x}_i^*; \mathbf{x}_{-i}^*) \geq \mathcal{V}_i(\mathbf{x}_i; \mathbf{x}_{-i}^*)$, for all $\mathbf{x}_i \in \vartheta_i(\mathbf{x}_{-i}^*)$, where $\mathcal{V}_i(\,\cdot\,;\,\cdot\,)$ is the utility function of the i-th player in a Fragile multi-CPR Game, given in (8).

In other words, $\mathbf{x}^* = (\mathbf{x}_1^*, \dots, \mathbf{x}_n^*) \in \mathcal{C}_n$ is a GNE if no player can increase her utility by unilaterally changing her strategy to any other element of the set $\vartheta_i(\mathbf{x}_{-i}^*)$. We may now proceed with stating our main results.

Theorem 2 *Consider a Fragile multi-CPR game, G, with $n \geq 1$ players and $m \geq 1$ CPRs, which satisfies Assumption 2. Then there exist constraint policies $\{\vartheta_i\}_{i\in[n]}$ such that the Constraint Fragile multi-CPR Game $(G, \{\vartheta_i\}_{i\in[n]})$ admits a Generalized Nash equilibrium.*

Given Theorem 2, it is natural to ask about the “size" of the set consisting of all GNEs of a Fragile multi-CPR Game. Let us note that it is a well known fact that Generalized Nash equilibrium problems tend to possess infinitely many GNEs [see Facchinei and Kanzow (2007, p. 192) and Dreves (2017)]. In the case of a single CPR, i.e., when $m = 1$, the corresponding Constrained Fragile CPR Game admits a unique GNE.

Theorem 3 *Consider a Fragile multi-CPR Game with $n \geq 1$ players and $m = 1$ CPR satisfying Assumption 2. Then the game admits a unique GNE.*

The proof of Theorem 3 is based upon a “first order condition" which is satisfied by the best response correspondence in a Fragile multi-CPR Game. It turns out that the aforementioned “first order condition" gives rise to *two types* of best responses for the players (see Theorem 8 below). In fact, we show that Theorem 3 is a consequence of a more general statement (i.e., Theorem 9 below) which provides an upper bound on the numbers of GNEs in a Fragile multi-CPR Game, subject to the assumption that best response of every player is of the first type.

For general m we are unable to determine the exact “size" of the set of GNEs. We conjecture its size is always finite. Our main result, which is valid when there are more players than CPRs, states that the set of GNEs is small in a measure-theoretic sense.

Theorem 4 *Consider a Fragile multi-CPR game, denoted G, with $n \geq 1$ players and $m \geq 1$ CPRs, which satisfies Assumption 2. Assume further that $m \leq n$, and let $\mathcal{N}(G)$ be the set consisting of all Generalized Nash equilibria of G. Then the $(n \cdot m)$-dimensional Lebesgue measure of $\mathcal{N}(G)$ is equal to zero.*

As mentioned already, and despite the fact that GNE problems tend to possess infinitely many solutions, we speculate that the “size" of the set $\mathcal{N}(G)$ in Theorem 4 can be reduced significantly.

Conjecture 1 The set $\mathcal{N}(G)$ is finite.

1.3 Brief outline of the proofs of main results

The proofs of our main results are inspired from the proof of Theorem 1, given in Hota et al. (2016). Having said that, it should also be mentioned that in a Fragile multi-CPR Game certain additional technicalities arise that are substantially different from those addressed in the proof of Theorem 1 in Hota et al. (2016). First and foremost, in a Fragile multi-CPR Game the strategy space of each player consists of m-dimensional vectors, a setting which requires concepts and ideas from multi-variable calculus.

In Hota et al. (2016) the existence of a Nash equilibrium in a Fragile CPR Game is established in two ways: the first approach employs Brouwer's fixed point theorem, and the second approach employs ideas from a particular class of games known as *Weak Strategic Substitute Games* [see Dubey et al. (2006)]. The first approach requires, among other things, the best response correspondence to be single-valued. The second approach requires the best-response correspondence to be decreasing. Both requirements may fail to hold true in a Fragile multi-CPR Game. Instead, we establish the existence of a Generalized Nash equilibrium for the Fragile multi-CPR Game by showing that it belongs to a particular class of "convex constrained games" which are known to possess Generalized Nash equilibria.

In Hota et al. (2016) the uniqueness of the Nash equilibrium for a Fragile CPR Game is established by showing that a particular auxiliary function, corresponding to the fact that the best response correspondence satisfies a particular "first order condition" [see Hota et al. (2016, Eq. (6), p. 142) for the precise formulation of the condition], is decreasing. Similar auxiliary functions are employed in the proofs of Theorems 3 and 4. However, the corresponding "first order conditions" are more delicate to characterise, and we do so by employing the KKT conditions to the optimization program corresponding to the best response correspondence (i.e., Problem (17) below). This allows to describe the best responses via a system of equations, having unique solution, and results in two types of "first order conditions" (see Theorem 8 below). Having established the first order conditions in a Fragile multi-CPR Game, we complete the proofs of our main results by employing monotonicity properties of certain auxiliary functions, in a way which may be seen as an extension of the approach taken in the proof of Theorem 1 in Hota et al. (2016).

1.4 Organization

The remaining part of our article is organised as follows. In Sect. 2 we show that the utility function of each player in a Fragile multi-CPR Game is concave on a particular subset of the strategy space. In Sect. 3 we prove Theorem 2, namely, we show that a Fragile multi-CPR Game admits a GNE. In Sect. 4 we show that the best response of each player in a Fragile multi-CPR Game satisfies certain "first order conditions", which are then used, in Sect. 5, in order to define suitable auxiliary functions whose monotonicity properties play a key role in the proofs of Theorems 3 and 4. Theorem 3 is proven in Sect. 6 and Theorem 4 is proven in Sect. 7. In Sect. 8 we show that a "restricted" version of a Fragile multi-CPR Game admits finitely many GNEs, a result which is then employed in order to formulate a conjecture which is equivalent to

Conjecture 1. Our paper ends with Sect. 9 which includes some concluding remarks and conjectures.

2 Concavity of utility function

In this section we show that the utility function, given by (8), of each player in a Fragile multi-CPR Games is concave in some particular subset of C_m. Before proceeding with the details let us mention that this particular subset will be used to define the constraint policies in the corresponding Constrained Fragile multi-CPR Game.

We begin with the following result, which readily follows from Hota et al. (2016, Lemma 1). Recall the definition of $\mathbf{x}_T^{(j)}$ and $\mathbf{x}_T^{j|i}$, given in (4), and the definition of the effective rate of return, $\mathcal{F}_{ij}$, given in (7).

Lemma 1 (see Hota et al. (2016, Lemma 1)) *Let $i \in [n]$ and $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$ be fixed. Then, for every $j \in [m]$, there exists a real number $\omega_{ij} \in (0, 1)$ such that $\mathcal{F}_{ij}(\mathbf{x}_T^{(j)}) > 0$, whenever $\mathbf{x}_T^{(j)} \in (0, \omega_{ij})$, and $\mathcal{F}_{ij}(\omega_{ij}) = 0$. Furthermore, provided that $\mathbf{x}_T^{j|i} < \omega_{ij}$, the function $\mathcal{E}_{ij}(\,\cdot\,; \mathbf{x}_T^{j|i})$ is concave in the interval $(0, \omega_{ij} - \mathbf{x}_T^{j|i})$.*

Proof We repeat the proof for the sake of completeness. Notice that $\mathcal{F}_{ij}(0) > 0$. Moreover, Assumption 2 implies that $\mathcal{F}_{ij}(1) < 0$. Since $\mathcal{F}_{ij}$ is continuous, the intermediate value theorem implies that there exists $\omega_{ij} \in (0, 1)$ such that $\mathcal{F}_{ij}(\omega_{ij}) = 0$. Since $\mathcal{F}_{ij}$ is assumed to be decreasing, the first statement follows, and we proceed with the proof of the second statement. To this end, notice that (6) yields

$$\begin{aligned}\frac{\partial^2}{\partial x_{ij}^2}\mathcal{E}_{ij}(x_{ij}; \mathbf{x}_T^{j|i}) &= a_i(a_i-1)x_{ij}^{a_i-2}\mathcal{F}_{ij}(\mathbf{x}_T^{(j)}) + 2a_i x_{ij}^{a_i-1}\frac{\partial}{\partial x_{ij}}\mathcal{F}_{ij}(\mathbf{x}_T^{(j)}) \\ &\quad + x_{ij}^{a_i}\frac{\partial^2}{\partial x_{ij}^2}\mathcal{F}_{ij}(\mathbf{x}_T^{(j)})\,.\end{aligned}$$

Observe that $\frac{\partial}{\partial x_{ij}}\mathcal{F}_{ij}(\mathbf{x}_T^{(j)}) = \frac{\partial}{\partial \mathbf{x}_T^{(j)}}\mathcal{F}_{ij}(\mathbf{x}_T^{(j)})$ and $\frac{\partial^2}{\partial x_{ij}^2}\mathcal{F}_{ij}(\mathbf{x}_T^{(j)}) = \frac{\partial^2}{\partial (\mathbf{x}_T^{(j)})^2}\mathcal{F}_{ij}(\mathbf{x}_T^{(j)})$. Moreover, Assumption 2 implies that $\frac{\partial^2}{\partial x_{ij}^2}\mathcal{F}_{ij}(\mathbf{x}_T^{(j)}), \frac{\partial}{\partial x_{ij}}\mathcal{F}_{ij}(\mathbf{x}_T^{(j)}) < 0$ as well as that $a_i - 1 \le 0$. Since $\mathcal{F}_{ij}(\mathbf{x}_T^{(j)}) > 0$ when $\mathbf{x}_T^{(j)} \in (0, \omega_{ij})$, we conclude that $\frac{\partial^2}{\partial x_{ij}^2}\mathcal{E}_{ij}(x_{ij}; \mathbf{x}_T^{j|i}) < 0$ and therefore $\mathcal{E}_{ij}(\,\cdot\,; \mathbf{x}_T^{j|i})$ is concave in the interval $(0, \omega_{ij} - \mathbf{x}_T^{j|i})$, as desired. □

Note that the first statement of Lemma 1 states that, given the choices of all remaining players, there is a player-specific investment-threshold, i.e., $\omega_{ij} - \mathbf{x}_T^{j|i}$, above which a player receives zero payoff. The second statement of Lemma 1 states that, given the choices of all players except player i, the utility of the i-th player from the j-th CPR is a concave function, when restricted on a particular interval. The coefficients ω_{ij} play a crucial role in the analysis since they will be used in order to define (see (12) below) constraint policies for the players in a Fragile multi-CPR Game that avoid "over-investments". The next result shows that an analogous statement holds true for the total utility of each player in a Fragile multi-CPR Game, namely, $\mathcal{V}_i(\mathbf{x}_i; \mathbf{x}_{-i})$, given by (8).

Given $i \in [n]$ and $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$, let

$$A(\mathbf{x}_{-i}) := \{j \in [m] : \mathbf{x}_T^{j|i} < \omega_{ij}\}, \tag{9}$$

where ω_{ij}, $j \in [m]$, is provided by Lemma 1. We refer to $A(\mathbf{x}_{-i})$ as the set of *active CPRs* corresponding to i and $\mathbf{x}_{-i}$.

Theorem 5 *Fix $i \in [n]$ and $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$. Let $A(\mathbf{x}_{-i})$ be the set of active CPRs corresponding to i and $\mathbf{x}_{-i}$, and consider the set $\mathcal{R}_{A(\mathbf{x}_{-i})} = \prod_{j \in A(\mathbf{x}_{-i})} (0, \omega_{ij} - \mathbf{x}_T^{j|i})$. Then the function $\mathcal{V}_{A(\mathbf{x}_{-i})} := \sum_{j \in A(\mathbf{x}_{-i})} \mathcal{E}_{ij}(x_{ij}; \mathbf{x}_T^{j\setminus i})$ is concave in $\mathcal{R}_{A(\mathbf{x}_{-i})}$.*

Proof If $|A(\mathbf{x}_{-i})| = 1$, then the result follows from Lemma 1 so we may assume that $|A(\mathbf{x}_{-i})| \geq 2$. The set $\mathcal{R}_{A(\mathbf{x}_{-i})}$ is clearly convex. Let $j, k \in A(\mathbf{x}_{-i})$ be such that $j \neq k$ and notice that

$$\frac{\partial^2 \mathcal{V}_{A(\mathbf{x}_{-i})}}{\partial x_{ij}\, \partial x_{ik}} = 0\,. \tag{10}$$

Moreover, by Lemma 1, we also have

$$\frac{\partial^2 \mathcal{V}_{A(\mathbf{x}_{-i})}}{\partial x_{ij}^2} = \frac{\partial^2 \mathcal{E}_{ij}(x_{ij}; \mathbf{x}_T^{j|i})}{\partial x_{ij}^2} < 0\,, \text{ for all } x_{ij} \in (0, \omega_{ij} - \mathbf{x}_T^{j|i})\,. \tag{11}$$

Given $\mathbf{x} \in \mathcal{R}_{A(\mathbf{x}_{-i})}$, denote by $H(\mathbf{x}) = \left(\frac{\partial^2 \mathcal{V}_{A(\mathbf{x}_{-i})}(\mathbf{x})}{\partial x_{ij}\, \partial x_{ik}}\right)_{j,k \in A(\mathbf{x}_{-i})}$ the Hessian matrix of $\mathcal{V}_{A(\mathbf{x}_{-i})}$ evaluated at $\mathbf{x}$, and let $\Delta_k(\mathbf{x})$, for $k \in A(\mathbf{x}_{-i})$, be the principal minors of $H(\mathbf{x})$ [see Berkovitz (2002, p. 111)]. Notice that (10) implies that $H(\mathbf{x})$ is a diagonal matrix. Therefore, using (11), it follows that $(-1)^k \cdot \Delta_k(\mathbf{x}) > 0$, when $\mathbf{x} \in \mathcal{R}_{A(\mathbf{x}_{-i})}$. In other words, $H(\cdot)$ is negative definite on the convex set $\mathcal{R}_{A(\mathbf{x}_{-i})}$ and we conclude [see Berkovitz (2002, Theorem 3.3, p. 110)] that $\mathcal{V}_{A(\mathbf{x}_{-i})}$ is concave in $\mathcal{R}_{A(\mathbf{x}_{-i})}$, as desired. □

3 Proof of Theorem 2: existence of GNE

In this section we show that the Fragile multi-CPR Game possesses a Generalized Nash equilibrium. Recall that the notion of Generalized Nash equilibrium depends upon the choice of constraint policies. Thus, before presenting the details of the proof, we first define the constrained policies under consideration.

Now fix $i \in [n]$ and $\mathbf{x}_{-i} = (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n) \in \mathcal{C}_{-i}$, where $\mathbf{x}_l = (x_{l1}, \ldots, x_{lm}) \in C_m$, for $l \in [n] \setminus \{i\}$. Recall that $\mathbf{x}_T^{j|i} = \sum_{\ell \in [n]\setminus\{i\}} x_{\ell j}$ and consider the set of active indices corresponding to i and $\mathbf{x}_{-i}$, i.e., consider the set $A(\mathbf{x}_{-i})$, defined in (9).

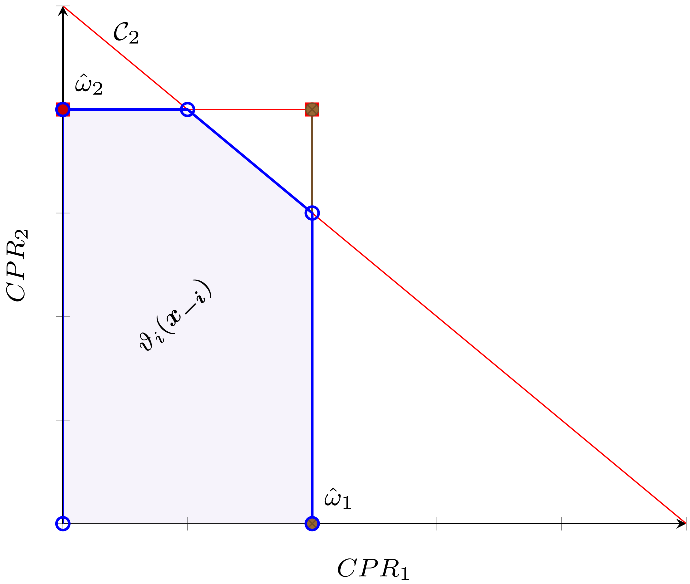

Fig. 1 Visualization of an instance of the constraint policy $\vartheta_i(\cdot)$ (blue shaded region) of player i in the case of $m = 2$, where we denote $\hat{\omega}_1 = \omega_{i1} - \mathbf{x}_T^{1|i}$ and $\hat{\omega}_2 = \omega_{i2} - \mathbf{x}_T^{2|i}$. (Color figure online)

Define the constraint policy $\vartheta_i(\cdot)$ that maps each element $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$ to the set

$$\vartheta_i(\mathbf{x}_{-i}) = C_m \bigcap \left\{ \prod_{j \in A(\mathbf{x}_{-i})} [0, \omega_{ij} - \mathbf{x}_T^{j|i}] \times \prod_{j \in [m] \setminus A(\mathbf{x}_{-i})} \{0\} \right\}, \tag{12}$$

where $\{\omega_{ij}\}_{j \in A(\mathbf{x}_{-i})}$ is given by Lemma 1.

As already mentioned, the definition of the constraint policy is taking into account the fact that players should avoid investing above the player-specific threshold provided by Lemma 1. Notice that, for every $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$, the set $\vartheta_i(\mathbf{x}_{-i})$ is *compact and convex*. Furthermore, since the vector all of whose coordinates are equal to zero belongs to the set $\vartheta_i(\mathbf{x}_{-i})$, it is *non-empty*. Figure 1 provides a visualization of the aforementioned constraint policy, in the case of $m = 2$.

We aim to show that the Constrained Fragile multi-CPR Game, with constraint policies given by (12), admits a Generalized Nash equilibrium. In order to do so, we employ the following theorem. Recall [see Ichiishi (1983, p. 32–33)] that a set-valued correspondence $\phi : X \to 2^Y$ is *upper semicontinuous* if for every open set $G \subset Y$, it holds that $\{x \in X : \phi(x) \subset G\}$ is an open set in X. A set-valued correspondence $\phi : X \to 2^Y$ is *lower semicontinuous* if every open set $G \subset Y$, it holds that $\{x \in X : \phi(x) \cap G \neq \emptyset\}$ is an open set in X. Recall also that, given $S \subset \mathbb{R}^s$, a function $f : S \to \mathbb{R}$ is quasi-concave if $f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\}$, for all $\mathbf{x} \neq \mathbf{y}$ in S and $\lambda \in (0, 1)$. Clearly, a concave function is also quasi-concave.

Theorem 6 *Let n players be characterized by strategy spaces X_i, $i \in [n]$, constraint policies ϕ_i, $i \in [n]$, and utility functions $\mathcal{V}_i : \prod_i X_i \to \mathbb{R}$, $i \in [n]$. Suppose further that the following hold true for every $i \in [n]$:*

1. *X_i is non-empty, compact, convex subset of a Euclidean space.*
2. *$\phi_i(\cdot)$ is both upper semicontinuous and lower semicontinuous in X_{-i}.*
3. *For all $\mathbf{x}_{-i} \in X_{-i}$, $\phi_i(\mathbf{x}_{-i})$ is nonempty, closed and convex.*
4. *$\mathcal{V}_i$ is continuous in $\prod_i X_i$.*
5. *For every $\mathbf{x}_{-i} \in X_{-i}$, the map $x_i \mapsto \mathcal{V}_i(x_i, \mathbf{x}_{-i})$ is quasi-concave on $\phi_i(\mathbf{x}_{-i})$.*

Then there exists a Generalized Nash equilibrium.

Proof This is a folklore result that can be found in various places. See, for example, Arrow and Debreu (1954), Facchinei and Kanzow (2007, Theorem 6), Ichiishi (1983, Theorem 4.3.1), Aubin (1998, Theorem 12.3), or Dutang (2013, Theorem 3.1). □

We are now ready to establish the existence of a GNE in the Constrained Fragile multi-CPR Game. In the following proof, $\|\cdot\|_d$ denotes d-dimensional Euclidean distance, and $B_d(\varepsilon) := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_d \leq \varepsilon\}$ is the closed ball of radius ε centered at the origin. Moreover, given $A \subset \mathbb{R}^d$ and $\varepsilon > 0$, we denote by $\{A\}_\varepsilon$ the set $A + B_d(\varepsilon) := \{a + b : a \in A \text{ and } b \in B_d(\varepsilon)\}$ and by $(1-\varepsilon)\cdot A$ the set $\{(1-\varepsilon)\cdot a : a \in A\}$.

Proof (Proof of Theorem 2) We apply Theorem 6. The strategy space of each player is equal to C_m, which is non-empty, compact and convex. Hence the first condition of Theorem 6 holds true. The third condition also holds true, by (12). Moreover, the fourth condition of Theorem 6 is immediate from the definition of utility, given in (8), while the fifth condition follows from Theorem 5.

It remains to show that the second condition of Theorem 6 holds true, i.e., that for each $i \in [n]$ the constrained policy $\vartheta_i(\cdot)$, given by (12), is both upper and lower semicontinuous. Towards this end, fix $i \in [n]$ and let $G \subset C_m$ be an open set. Consider the sets

$$G^+ := \{\mathbf{x}_{-i} \in \mathcal{C}_{-i} : \vartheta_i(\mathbf{x}_{-i}) \subset G\} \quad \text{and} \quad G^- := \{\mathbf{x}_{-i} \in \mathcal{C}_{-i} : \vartheta_i(\mathbf{x}_{-i}) \cap G \neq \emptyset\}.$$

We have to show that both G^+ and G^- are open subsets of $\mathcal{C}_{-i}$. We first show that G^+ is open.

If G^+ is empty then the result is clearly true, so we may assume that $G^+ \neq \emptyset$. Let $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_{i-1}, \mathbf{y}_{i+1}, \ldots, \mathbf{y}_n) \in G^+$; hence $\vartheta_i(\mathbf{y}) \subset G$. We have to show that there exists $\varepsilon > 0$ such that for every $\mathbf{x} \in \mathcal{C}_{-i}$ with $\|\mathbf{x} - \mathbf{y}\|_{(n-1)m} < \varepsilon$, we have $\vartheta_i(\mathbf{x}) \subset G$. Since $\vartheta_i(\mathbf{y})$ is a compact subset of the open set G, it follows that there exists $\varepsilon_0 > 0$ such that $\{\vartheta_i(\mathbf{y})\}_{\varepsilon_0} \subset G$. Since summation is continuous, there exists $\varepsilon_1 > 0$ such that for every $\mathbf{x} \in \mathcal{C}_{-i}$ with $\|\mathbf{x} - \mathbf{y}\|_{(n-1)m} < \varepsilon_1$ it holds $\mathbf{x} \in \{\vartheta_i(\mathbf{y})\}_{\varepsilon_0}$. The desired ε is given by ε_1. Hence G^+ is an open set, and we proceed with showing that G^- is open as well.

We may assume that G^- is non-empty. For each $i \in [n]$, let $g_i : \mathcal{C}_{-i} \to \mathbb{R}^m_{\geq 0}$ be the continuous function whose j-th coordinate, for $j \in [m]$, is given by

$$g_{ij}(\mathbf{x}_{-i}) = \begin{cases} \omega_{ij} - \mathbf{x}_T^{j|i}, & \text{if } \omega_{ij} - \mathbf{x}_T^{j|i} > 0 \\ 0, & \text{if } \omega_{ij} - \mathbf{x}_T^{j|i} \leq 0, \end{cases}$$

where ω_{ij} is given by Lemma 1. Let $h : \mathbb{R}^m_{\geq 0} \to 2^{C_m}$ be the set-valued function defined by $h(z_1, \dots, z_m) = \prod_{j \in [m]} [0, z_j]$, with the convention $[0, 0] := \{0\}$. Clearly, it holds that $\vartheta_i = h \circ g_i$, for all $i \in [n]$.

We claim that h is lower semicontinuous. If the claim holds true then it follows that the set $H := \{\mathbf{z} \in \mathbb{R}^m_{\geq 0} : h(\mathbf{z}) \cap G \neq \emptyset\}$ is open. Notice that $G^- \neq \emptyset$ implies that $H \neq \emptyset$. Since g_i is continuous, it follows that the preimage of H under g_i, i.e., $g_i^{-1}(H)$, is open. In other words, the set $\{\mathbf{x} \in \mathcal{C}_{-i} : h \circ g_i(\mathbf{x}) \cap G \neq \emptyset\} = \{\mathbf{x} \in \mathcal{C}_{-i} : \vartheta_i(\mathbf{x}) \cap G \neq \emptyset\}$ is open and the proof of the theorem is complete.

It remains to prove the claim, i.e., that h is lower semicontinuous. To this end, let $G \subset C_m$ be an open set, and let $G^* := \{\mathbf{z} \in \mathbb{R}^m_{\geq 0} : h(\mathbf{z}) \cap G \neq \emptyset\}$. We have to show that G^* is open; that is, we have to show that for every $\mathbf{z} \in G^*$ there exists $\varepsilon > 0$ such that $\mathbf{w} \in G^*$, for all $\mathbf{w}$ with $\|\mathbf{z} - \mathbf{w}\|_m < \varepsilon$. Fix $\mathbf{z} \in G^*$. Since $h(\mathbf{z})$ is compact and G is open, it follows that there exists $\varepsilon_0 > 0$ such that $(1 - \varepsilon_0) \cdot h(\mathbf{z}) \cap G \neq \emptyset$. Now choose $\varepsilon > 0$ such that for every $\mathbf{w} \in C_m$ for which $\|\mathbf{z} - \mathbf{w}\|_m < \varepsilon$ it holds $(1 - \varepsilon_0) \cdot h(\mathbf{z}) \subset h(\mathbf{w})$. In other words, for this particular choice of $\varepsilon > 0$ it holds $h(\mathbf{w}) \cap G \neq \emptyset$, for every $\mathbf{w}$ with $\|\mathbf{z} - \mathbf{w}\|_m < \varepsilon$. The claim follows. □

4 Best response correspondence

Having established the existence of a GNE for a Fragile multi-CPR Game, we now proceed with the proofs of Theorems 3 and 4. The proofs will be obtained in two steps. In the first step we deduce certain "first order conditions" which are satisfied by the best response correspondence of each player in the game. In the second step we employ the first order conditions in order to define certain auxiliary functions, whose monotonicity will be employed in the proofs of the aforementioned theorems. In this section we collect some results pertaining to the first step. We begin with recalling the notion of the best response correspondence [see Laraki et al. (2019)].

Given $i \in [n]$ and $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$, let $\vartheta_i(\cdot)$ denote the constraint policy given by (12), and consider the *best response* of the i-th player in the Fragile multi-CPR Game defined as follows:

$$B_i(\mathbf{x}_{-i}) = \arg \max_{\mathbf{x}_i \in \vartheta_i(\mathbf{x}_{-i})} \mathcal{V}_i(\mathbf{x}_i; \mathbf{x}_{-i}) \,, \tag{13}$$

where $\mathcal{V}_i$ is the utility of the i-th player, given by (8). Notice that $B_i(\cdot)$ is a correspondence $B_i : \mathcal{C}_{-i} \to 2^{C_m}$, where 2^{C_m} denotes the class consisting of all subsets of C_m. For $j \in [m]$, we denote by $B_{ij}(\mathbf{x}_{-i})$ the j-th component of $B_i(\mathbf{x}_{-i})$; hence we have

$$B_i(\mathbf{x}_{-i}) = (B_{i1}(\mathbf{x}_{-i}), \dots, B_{im}(\mathbf{x}_{-i})) \,.$$

Remark 1 Notice that Definition 2 implies that if $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathcal{C}_n$ is a GNE of a Constrained Fragile multi-CPR Game, with constraint policies given by (12), then for each $i \in [n]$ it holds $\mathbf{x}_i \in B_i(\mathbf{x}_{-i})$.

Recall that $A(\mathbf{x}_{-i})$ denotes the set of active CPRs corresponding to $\mathbf{x}_{-i}$, defined in (9), and notice that $B_{ij}(\mathbf{x}_{-i}) = 0$, for all $j \in [m] \setminus A(\mathbf{x}_{-i})$.

For $x_{ij} \in [0, 1]$, let $\psi_{ij}(x_{ij}; \mathbf{x}_T^{j|i})$ be the function defined via

$$\psi_{ij}(x_{ij}; \mathbf{x}_T^{j|i}) = x_{ij} \cdot \frac{\partial}{\partial x_{ij}} \mathcal{F}_{ij}(x_{ij} + \mathbf{x}_T^{j|i}) + a_i \mathcal{F}_{ij}(x_{ij} + \mathbf{x}_T^{j|i}) . \quad (14)$$

Lemma 2 *Fix $i \in [n]$ and $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$ and let $\mathcal{R}_{A(\mathbf{x}_{-i})} = \prod_{j \in A(\mathbf{x}_{-i})} (0, \omega_{ij} - \mathbf{x}_T^{j|i})$, where ω_{ij} is provided by Lemma 1. Then a global maximum of the function $\mathcal{V}_{\mathbf{x}_{-i}} := \sum_{j \in A(\mathbf{x}_{-i})} \mathcal{E}_{ij}(x_{ij}; \mathbf{x}_T^{j|i})$ defined on the set $\mathcal{R}_{A(\mathbf{x}_{-i})}$ is given by the unique solution of the following system of equations:*

$$\psi_{ij}(x_{ij}; \mathbf{x}_T^{j|i}) = 0, \ \ \textit{for} \ \ j \in A(\mathbf{x}_{-i}) , \quad (15)$$

where $\psi_{ij}(x_{ij}; \mathbf{x}_T^{j|i})$ is defined in (14).

Proof To simplify notation, we write $\psi_{ij}(\cdot)$ instead of $\psi_{ij}(\,\cdot\,; \mathbf{x}_T^{j|i})$. Using (6) and (8), it is straightforward to verify that for every $j \in A(\mathbf{x}_{-i})$ it holds

$$\frac{\partial \mathcal{V}_{\mathbf{x}_{-i}}}{\partial x_{ij}} = \frac{\partial \mathcal{E}_{ij}(x_{ij}; \mathbf{x}_T^{j|i})}{\partial x_{ij}} = x_{ij}^{a_i - 1} \cdot \psi_{ij}(x_{ij}) . \quad (16)$$

Now notice that $\psi_{ij}(0) > 0$ as well as $\psi_{ij}(\omega_{ij} - \mathbf{x}_T^{j|i}) < 0$. Moreover, Assumption 2 readily implies that $\psi_{ij}(\cdot)$ is strictly decreasing on the interval $(0, \omega_{ij} - \mathbf{x}_T^{j|i})$. The intermediate value theorem implies that there exists unique $\lambda_{ij} \in (0, \omega_{ij} - \mathbf{x}_T^{j|i})$ such that $\psi_{ij}(\lambda_{ij}) = 0$. Hence, it follows from (16) that the points λ_{ij}, for $j \in A(\mathbf{x}_{-i})$, are critical points of the function $\mathcal{V}_{\mathbf{x}_{-i}}$, which is concave on the open and convex set $\mathcal{R}_{A(\mathbf{x}_{-i})}$, by Theorem 5. It follows [see Berkovitz (2002, Theorem 2.4, p. 132)] that $\{\lambda_{ij}\}_{j \in A(\mathbf{x}_{-i})}$ is a global maximum of $\mathcal{V}_{\mathbf{x}_{-i}}$ on $\mathcal{R}_{A(\mathbf{x}_{-i})}$. We conclude that $\mathcal{V}_{\mathbf{x}_{-i}}$ is maximized when $x_{ij} = \lambda_{ij}$, for $j \in A(\mathbf{x}_{-i})$, as desired. □

Remark 2 Let us remark that the solution of the system of equations given by (15) may not belong to the set C_m. More precisely, it could happen that the solution of the system of equations (15), say $\{\lambda_{ij}\}_{j \in A(\mathbf{x}_{-i})}$, satisfies $\sum_{j \in A(\mathbf{x}_{-i})} \lambda_{ij} > 1$. This is a crucial difference between the Fragile CPR Game and the Fragile multi-CPR Game.

Now notice that, given $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$, the best response of player i is a local maximum of the following program:

$$\begin{aligned} \underset{\{x_{ij}\}_{j \in A(\mathbf{x}_{-i})}}{\text{maximize}} \quad & \mathcal{V}_{\mathbf{x}_{-i}} := \sum_{j \in A(\mathbf{x}_{-i})} \mathcal{E}_{ij}(x_{ij}; \mathbf{x}_T^{j|i}) \\ \text{subject to} \quad & \sum_{j \in A(\mathbf{x}_{-i})} x_{ij} \leq 1 \\ & 0 \leq x_{ij} \leq \omega_{ij} - \mathbf{x}_T^{j|i}, \ \text{for all} \ \ j \in A(\mathbf{x}_{-i}) . \end{aligned}$$

Equivalently, the best response of player i is a local minimum of the following program:

$$\begin{aligned} \underset{\{x_{ij}\}_{j\in A(\mathbf{x}_{-i})}}{\text{minimize}} \quad & -\sum_{j\in A(\mathbf{x}_{-i})} \mathcal{E}_{ij}(x_{ij};\mathbf{x}_T^{j|i}) \\ \text{subject to} \quad & \sum_{j\in A(\mathbf{x}_{-i})} x_{ij} \le 1 \\ & 0 \le x_{ij} \le \omega_{ij} - \mathbf{x}_T^{j|i}, \text{ for all } j \in A(\mathbf{x}_{-i})\,. \end{aligned} \tag{17}$$

Notice that since $\mathcal{E}_{ij}(\,\cdot\,;\mathbf{x}_T^{j|i})$ is concave on $(0, \omega_{ij} - \mathbf{x}_T^{j|i})$, by Lemma 1, it follows that Problem (17) is a separable convex knapsack program [see Levi et al. (2014) and Stefanov (2015)]. We are going to describe the optima of Problem (17) using the KKT conditions. The KKT conditions pertain to the Lagrangian corresponding to Problem (17), which is defined as the following quantity:

$$\begin{aligned} \mathcal{L} := -\mathcal{V}_{\mathbf{x}_{-i}} + \kappa_0 \cdot \left(\sum_{j\in A(\mathbf{x}_{-i})} x_{ij} - 1 \right) + \sum_{j\in A(\mathbf{x}_{-i})} \mu_j \cdot (x_{ij} + \mathbf{x}_T^{j|i} - \omega_{ij}) \\ + \sum_{j\in A(\mathbf{x}_{-i})} \nu_j \cdot (-x_{ij}), \end{aligned}$$

where κ_0, $\{\mu_j\}_j$, $\{\nu_j\}_j$ are real numbers. The KKT conditions corresponding to problem (17) read as follows [(see Luptacik (2010, Theorem 3.8)].

Theorem 7 (KKT conditions for Problem (17)) *If $\{x_{ij}\}_{j\in A(\mathbf{x}_{-i})}$ is a local minimum of Problem (17), then there exist* non-negative *real numbers κ_0, $\{\mu_j\}_{j\in A(\mathbf{x}_{-i})}$, and $\{\nu_j\}_{j\in A(\mathbf{x}_{-i})}$ such that:*

1. *For all $j \in A(\mathbf{x}_{-i})$ it holds $-x_{ij}^{a_i-1} \cdot \psi_{ij}(x_{ij};\mathbf{x}_T^{j|i}) + \kappa_0 + \mu_j - \nu_j = 0$, where ψ_{ij} is given by (14).*
2. $\kappa_0 \cdot \left(\sum_{j\in A(\mathbf{x}_{-i})} x_{ij} - 1 \right) = 0.$
3. *$\mu_j \cdot (x_{ij} + \mathbf{x}_T^{j|i} - \omega_{ij}) = 0$, for all $j \in A(\mathbf{x}_{-i})$.*
4. *$\nu_j \cdot x_{ij} = 0$, for all $j \in A(\mathbf{x}_{-i})$.*
5. *$0 \le x_{ij} \le \omega_{ij} - \mathbf{x}_T^{j|i}$, for all $j \in A(\mathbf{x}_{-i})$.*

We aim to employ Theorem 7 in order to describe a local minimum of Problem (17) via the solution of a system of equations. This will require the following result, which is presumably reported somewhere in the literature but, lacking a reference, we include a proof for the sake of completeness.

Lemma 3 *Fix a positive integer s and, for each $j \in [s]$, let $f_j : \mathbb{R} \to \mathbb{R}$ be a strictly decreasing function. Then there exists at most one vector $(c, x_1, \ldots, x_s) \in \mathbb{R}^{s+1}$ such that*

$$f_j(x_j) = c, \;\; \textit{for all} \;\; j \in [s], \;\; \textit{and} \;\; \sum_{j\in[s]} x_j = 1\,.$$

Proof The proof is deferred to "Appendix B". □

We may now proceed with describing the best responses of each player in the Fragile multi-CPR Game via a system of "first order conditions".

Theorem 8 *Let $i \in [n]$ and $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$ be fixed. Suppose that $\{x_{ij}\}_{j\in A(\mathbf{x}_{-i})}$ is a best response of player i in the Fragile multi-CPR Game. Then $\{x_{ij}\}_{j\in A(\mathbf{x}_{-i})}$ is either of the following two types:*

- ***Type I:*** *There exists $J_{\mathbf{x}_{-i}} \subset A(\mathbf{x}_{-i})$ such that $x_{ij} = 0$, when $j \in A(\mathbf{x}_{-i}) \setminus J_{\mathbf{x}_{-i}}$, and $\{x_{ij}\}_{j\in J_{\mathbf{x}_{-i}}}$ satisfy the following inequality, and are given by the unique solution of the following system of equations:*

$$\sum_{j\in J_{\mathbf{x}_{-i}}} x_{ij} < 1 \quad \textit{and} \quad \psi_{ij}(x_{ij}; \mathbf{x}_T^{j|i}) = 0, \ \textit{for } j \in J_{\mathbf{x}_{-i}},$$

  where $\psi_{ij}(\,\cdot\,; \mathbf{x}_T^{j|i})$ is defined in (14).
- ***Type II:*** *There exists $J_{\mathbf{x}_{-i}} \subset A(\mathbf{x}_{-i})$ and a real number $\kappa_0 \geq 0$ such that $x_{ij} = 0$, when $j \in A(\mathbf{x}_{-i}) \setminus J_{\mathbf{x}_{-i}}$, and $\{x_{ij}\}_{j\in J}$ are given by the unique solution of the following system of equations:*

$$\sum_{j\in J_{\mathbf{x}_{-i}}} x_{ij} = 1 \quad \textit{and} \quad x_{ij}^{a_i-1} \cdot \psi_{ij}(x_{ij}; \mathbf{x}_T^{j|i}) = \kappa_0, \ \textit{for } j \in J_{\mathbf{x}_{-i}},$$

  where $\psi_{ij}(\,\cdot\,; \mathbf{x}_T^{j|i})$ is defined in (14).

Proof Let $\{x_{ij}\}_{j\in A(\mathbf{x}_{-i})}$ be a best response of player $i \in [n]$. Then $\{x_{ij}\}_{j\in A(\mathbf{x}_{-i})}$ is a local minimum of Problem (17); hence it satisfies the KKT Conditions of Theorem 7.

If $x_{ij} = \omega_{ij} - \mathbf{x}_T^{j|i}$, for some $j \in A(\mathbf{x}_{-i})$, then Lemma 1 and (6) imply that $\mathcal{E}_{ij}(x_{ij}; \mathbf{x}_T^{j|i}) = 0$. Hence player i could achieve the same utility from the j-th CPR by choosing $x_{ij} = 0$. Thus we may assume that $x_{ij} < \omega_{ij} - \mathbf{x}_T^{j|i}$, for all $j \in A(\mathbf{x}_{-i})$ and therefore Theorem 7.(3) implies that $\mu_j = 0$, for all $j \in A(\mathbf{x}_{-i})$. Now let

$$J_{\mathbf{x}_{-i}} = \{j \in A(\mathbf{x}_{-i}) : x_{ij} \neq 0\}, \tag{18}$$

and notice that Theorem 7.(4) implies that $\nu_j = 0$ for $j \in J_{\mathbf{x}_{-i}}$. We distinguish two cases.

Suppose first that $\sum_{j\in J_{\mathbf{x}_{-i}}} x_{ij} < 1$. Then Theorem 7.(2) yields $\kappa_0 = 0$, and therefore Theorem 7.(1) implies that $\{x_{ij}\}_{i\in J_{\mathbf{x}_{-i}}}$ is given by the unique solution of the following system of equations:

$$\psi_{ij}(x_{ij}; \mathbf{x}_T^{j|i}) = 0, \ \text{ for } j \in J_{\mathbf{x}_{-i}}.$$

In other words, if $\sum_{j\in J_{\mathbf{x}_{-i}}} x_{ij} < 1$ then $\{x_{ij}\}_{j\in A(\mathbf{x}_{-i})}$ is of Type I.

Now assume that $\sum_{j\in J_{\mathbf{x}_{-i}}} x_{ij} = 1$. Then Theorems 7.(1) and 7.(2) imply that there exists $\kappa_0 \geq 0$ such that $-x_{ij}^{a_i-1} \cdot \psi_{ij}(x_{ij}; \mathbf{x}_T^{j|i}) = -\kappa_0$, for all $j \in J_{\mathbf{x}_{-i}}$. In other words, $\{x_{ij}\}_{j\in J_{\mathbf{x}_{-i}}}$ and κ_0 are given by the solution of the following system of equations:

$$\sum_{j\in J_{\mathbf{x}_{-i}}} x_{ij} = 1 \text{ and } x_{ij}^{a_i-1} \cdot \psi_{ij}(x_{ij}; \mathbf{x}_T^{j|i}) = \kappa_0, \text{ for all } j \in J_{\mathbf{x}_{-i}}. \tag{19}$$

Since the functions $f_{ij}(x_{ij}) := x_{ij}^{a_i-1} \cdot \psi_{ij}(x_{ij}; \mathbf{x}_T^{j|i})$, for $j \in J_{\mathbf{x}_{-i}}$, are strictly decreasing, Lemma 3 implies that the system of equations in (19) has a unique solution. Hence $\{x_{ij}\}_{j\in A(\mathbf{x}_{-i})}$ is of Type II and the result follows. □

We refer to the set $J_{\mathbf{x}_{-i}}$ provided by Theorem 8, defined in (18), as the set of *effective* CPRs corresponding to $i \in [n]$ and $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$. In the next section we employ Theorem 8 in order to define certain auxiliary functions (i.e., (24) and (25) below) whose monotonicity will play a key role in the proof of Theorem 4.

Let us remark that Theorem 8 roughly states that, given a strategy profile of the other players, a dominant strategy of a player in a Fragile multi-CPR game is to either maximize her utility in as many CPRs as possible, or to invests all her endowment in such a way that the slopes of her utilities from each effective CPR are equal. Since a GNE, say $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{C}_n$, is a point for which every player has chosen a best response, by Remark 1, it follows that every $\mathbf{x}_i$ is either of Type I or of Type II. In other words, the GNEs of a Fragile multi-CPR Game can be found among the solutions of the system of equations provided by Theorem 8.

5 Auxiliary functions

In this section we define and state basic properties of certain auxiliary functions, whose monotonicity will be used in the proofs of Theorems 3 and 4, and whose definition depends upon the "first order conditions" provided by Theorem 8.

Let us begin with some notation and remarks. Fix $i \in [n]$ and $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$, and recall from (13) that $B_i(\mathbf{x}_{-i})$ denotes a best response of player i and that $B_{ij}(\mathbf{x}_{-i})$ is its j-th component. To simplify notation, let us denote $b_{ij} := B_{ij}(\mathbf{x}_{-i})$. From Theorem 8 we know that there exists $J_{\mathbf{x}_{-i}} \subset A(\mathbf{x}_{-i})$ such that $b_{ij} = 0$, for $j \in A(\mathbf{x}_{-i}) \setminus J_{\mathbf{x}_{-i}}$, and either

$$\sum_{j\in J_{\mathbf{x}_{-i}}} b_{ij} < 1 \text{ and } \psi_{ij}(b_{ij}; \mathbf{x}_{-i}) = 0, \text{ for all } j \in J_{\mathbf{x}_{-i}}, \tag{20}$$

or

$$\sum_{j\in J_{\mathbf{x}_{-i}}} b_{ij} = 1 \text{ and } b_{ij}^{a_i-1} \cdot \psi_{ij}(b_{ij}; \mathbf{x}_{-i}) = \kappa_0, \text{ for all } j \in J_{\mathbf{x}_{-i}} \text{ and some } \kappa_0 \geq 0. \tag{21}$$

In particular, it holds $b_{ij} > 0$, for all $j \in J_{\mathbf{x}_{-i}}$. Using (14), it follows that the second statement of (20) is equivalent to

$$b_{ij} \cdot \frac{\partial}{\partial x_{ij}} \mathcal{F}_{ij}(b_{ij} + \mathbf{x}_T^{j|i}) + a_i \mathcal{F}_{ij}(b_{ij} + \mathbf{x}_T^{j|i}) = 0, \quad \text{for all } j \in J_{\mathbf{x}_{-i}}, \tag{22}$$

and that the second statement of (21) is equivalent to

$$b_{ij}^{a_i - 1} \cdot \left(b_{ij} \cdot \frac{\partial}{\partial x_{ij}} \mathcal{F}_{ij}(b_{ij} + \mathbf{x}_T^{j|i}) + a_i \mathcal{F}_{ij}(b_{ij} + \mathbf{x}_T^{j|i}) \right) = \kappa_0, \quad \text{for all } j \in J_{\mathbf{x}_{-i}}. \tag{23}$$

Now, given $\mathbf{x}_{-i} \in \mathcal{C}_{-i}$, $j \in J_{\mathbf{x}_{-i}}$ and $\kappa_0 \geq 0$, define for each $i \in [n]$ the functions

$$\mathcal{G}_{ij}(x_{ij} + \mathbf{x}_T^{j|i}) := -\frac{a_i \mathcal{F}_{ij}(x_{ij} + \mathbf{x}_T^{j|i})}{\frac{\partial}{\partial x_{ij}} \mathcal{F}_{ij}(x_{ij} + \mathbf{x}_T^{j|i})}, \text{ for } x_{ij} \in (0, \omega_{ij} - \mathbf{x}_T^{j|i}) \tag{24}$$

and

$$\mathcal{H}_{ij}(x_{ij} + \mathbf{x}_T^{j|i}; \kappa_0) := -\frac{a_i \mathcal{F}_{ij}(x_{ij} + \mathbf{x}_T^{j|i})}{\frac{-\kappa_0}{x_{ij}^{a_i}} + \frac{\partial}{\partial x_{ij}} \mathcal{F}_{ij}(x_{ij} + \mathbf{x}_T^{j|i})}, \text{ for } x_{ij} \in (0, \omega_{ij} - \mathbf{x}_T^{j|i}). \tag{25}$$

Notice that Assumption 2 implies that the denominators in (24) and (25) are non-zero, and thus the functions are well-defined. Notice also that (22) implies that when b_{ij} is of Type I it holds

$$\mathcal{G}_{ij}(b_{ij} + \mathbf{x}_T^{j|i}) = b_{ij}, \tag{26}$$

while (23) implies that when b_{ij} is of Type II it holds

$$\mathcal{H}_{ij}(b_{ij} + \mathbf{x}_T^{j|i}; \kappa_0) = b_{ij}, . \tag{27}$$

Observe also that it holds $\mathcal{G}_{ij}(x_{ij} + \mathbf{x}_T^{j|i}) \geq \mathcal{H}_{ij}(x_{ij} + \mathbf{x}_T^{j|i}; \kappa_0)$, for all $x_{ij} \in [0, \omega_{ij} - \mathbf{x}_T^{j|i}]$. Let us, for future reference, collect a couple of observations about the functions $\mathcal{G}_{ij}, \mathcal{H}_{ij}$.

Lemma 4 *Let $i \in [n]$ and $j \in [m]$ be fixed. Then the functions $\mathcal{G}_{ij}(\cdot)$ and $\mathcal{H}_{ij}(\cdot\,; \kappa_0)$, defined in (24) and (25) respectively, are strictly decreasing in the interval $[0, \omega_{ij}]$.*

Proof The, rather straightforward, proof is deferred to "Appendix C". □

6 Proof of Theorem 3

In this section we prove Theorem 3. We begin with some notation. Consider a GNE, say $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{C}_n$, where $\mathbf{x}_i = (x_{i1}, \ldots, x_{im}) \in C_m$, of a Fragile multi-CPR

Game satisfying Assumption 2. Given $j \in [m]$, let

$$\mathcal{S}(\mathbf{x}_T^{(j)}) = \{i \in [n] : \mathbf{x}_T^{(j)} < \omega_{ij} \text{ and } x_{ij} > 0\} \tag{28}$$

be the *support* of the j-th CPR and let

$$\mathcal{S}_I(\mathbf{x}_T^{(j)}) = \{i \in \mathcal{S}(\mathbf{x}_T^{(j)}) : \mathbf{x}_i \text{ is of Type I}\} \tag{29}$$

be the *support of Type I*, consisting of those players in the support of the j-th CPR whose best response is of Type I, and

$$\mathcal{S}_{II}(\mathbf{x}_T^{(j)}) = \{i \in \mathcal{S}(\mathbf{x}_T^{(j)}) : \mathbf{x}_i \text{ is of Type II}\} \tag{30}$$

be the *support of Type II*, consisting of those players in the support of the j-th CPR whose best response is of Type II. Clearly, in view of Theorem 8, it holds $\mathcal{S}(\mathbf{x}_T^{(j)}) = \mathcal{S}_I(\mathbf{x}_T^{(j)}) \cup \mathcal{S}_{II}(\mathbf{x}_T^{(j)})$.

We employ the properties of the auxiliary functions in the proof of Theorem 3, a basic ingredient of which is the fact that in the setting of Theorem 3 the support of Type II is empty. The proof is similar to the proof of Theorem 1, given in Hota et al. (2016, p. 155). In fact, we prove a bit more. We show that Theorem 3 is a consequence of the following result.

Theorem 9 *Consider a Fragile multi-CPR Game with $n \geq 1$ players and $m \geq 1$ CPRs satisfying Assumption 2. Then there exists at most one GNE $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ for which $\mathbf{x}_i$ is of Type I, for all $i \in [n]$.*

Proof Let $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ be a GNE such that $\mathbf{x}_i$ is of Type I, for all $i \in [n]$ and note that, since $\mathbf{x}_i$ is of Type I, it holds $\sum_j x_{ij} < 1$, for all $i \in [n]$. For each $j \in [m]$, let $\mathcal{S}_0(\mathbf{x}_T^{(j)}) := \{i \in [n] : \mathbf{x}_T^{(j)} < \omega_{ij}\}$. We claim that $\mathcal{S}_0(\mathbf{x}_T^{(j)}) = \mathcal{S}(\mathbf{x}_T^{(j)})$. Indeed, if there exists $i \in \mathcal{S}_0(\mathbf{x}_T^{(j)}) \setminus \mathcal{S}(\mathbf{x}_T^{(j)})$ then $x_{ij} = 0$ and since it holds $\mathbf{x}_T^{(j)} < \omega_{ij}$ and $\sum_j x_{ij} < 1$, it follows that player i could increase her utility by investing a suitably small amount, say $\varepsilon > 0$, in the j-th CPR. But then this implies that $\mathbf{x}$ cannot be a GNE, a contradiction. Hence $\mathcal{S}_0(\mathbf{x}_T^{(j)}) = \mathcal{S}(\mathbf{x}_T^{(j)})$.

We claim that for any two distinct GNEs, say $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ and $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_n)$, for which $\mathbf{x}_i, \mathbf{y}_i$ are of Type I for all $i \in [n]$, it holds $\mathbf{x}_T^{(j)} = \mathbf{y}_T^{(j)}$, for all $j \in [m]$. Indeed, if the claim is not true, then there exists $j \in [m]$ such that $\mathbf{x}_T^{(j)} \neq \mathbf{y}_T^{(j)}$. Suppose, without loss of generality, that $\mathbf{x}_T^{(j)} < \mathbf{y}_T^{(j)}$.

Since $\mathbf{x}_i$ is of Type I, for all $i \in [n]$, it follows that $\mathcal{S}(\mathbf{x}_T^{(j)}) = \mathcal{S}_I(\mathbf{x}_T^{(j)})$ and $\mathcal{S}(\mathbf{y}_T^{(j)}) = \mathcal{S}_I(\mathbf{y}_T^{(j)})$. Moreover, since $\mathbf{x}_T^{(j)} < \mathbf{y}_T^{(j)}$ it follows that $\mathcal{S}_I(\mathbf{y}_T^{(j)}) = \mathcal{S}_0(\mathbf{y}_T^{(j)}) \subset \mathcal{S}_0(\mathbf{x}_T^{(j)}) = \mathcal{S}_I(\mathbf{x}_T^{(j)})$.

Now notice that (26) implies that $\mathcal{G}_{ij}(\mathbf{x}_T^{(j)}) = x_{ij}$, for all $i \in \mathcal{S}_I(\mathbf{x}_T^{(j)})$, and $\mathcal{G}_{ij}(\mathbf{y}_T^{(j)}) = y_{ij}$, for all $i \in \mathcal{S}_I(\mathbf{y}_T^{(j)})$. Since $\mathcal{S}_I(\mathbf{y}_T^{(j)}) \subset \mathcal{S}_I(\mathbf{x}_T^{(j)})$ it holds that

$$\sum_{i \in \mathcal{S}_I(\mathbf{y}_T^{(j)})} \mathcal{G}_{ij}(\mathbf{x}_T^{(j)}) \leq \mathbf{x}_T^{(j)} < \mathbf{y}_T^{(j)} = \sum_{i \in \mathcal{S}_I(\mathbf{y}_T^{(j)})} \mathcal{G}_{ij}(\mathbf{y}_T^{(j)}) . \tag{31}$$

However, since $\mathcal{G}_{ij}$ is strictly decreasing, it follows that $\mathcal{G}_{ij}(\mathbf{x}_T^{(j)}) > \mathcal{G}_{ij}(\mathbf{y}_T^{(j)})$, for all $i \in \mathcal{S}_I(\mathbf{y}_T^{(j)})$, which contradicts (31). We conclude that $\mathbf{x}_T^{(j)} = \mathbf{y}_T^{(j)}$ and $\mathcal{S}_I(\mathbf{x}_T^{(j)}) = \mathcal{S}_I(\mathbf{y}_T^{(j)})$. Finally, given a total investment $\mathbf{x}$ of the players at a GNE, we claim that the optimal investment of every player on any CPR is unique. Indeed, if a player, say $i \in [n]$, has two optimal investments, say $x < z$, on the j-th CPR, then it holds $\mathcal{G}_{ij}(\mathbf{x}) = x < z = \mathcal{G}_{ij}(\mathbf{x})$, a contradiction. The result follows. □

Theorem 3 is a direct consequence of Theorem 9, as we now show.

Proof (Proof of Theorem 3) We know from Theorem 2 that the game admits a GNE, and it is therefore enough to show that it is unique. Since $m = 1$, the first condition in Assumption 2 implies that no player invests an amount of 1 in the CPR which in turn implies that all coordinates of any GNE are of Type I. The result follows from Theorem 9. □

Observe that a basic ingredient in the proof of Theorem 9 is the fact that $\mathcal{S}(\mathbf{x}_T) = \mathcal{S}_I(\mathbf{x}_T)$, $\mathcal{S}(\mathbf{y}_T) = \mathcal{S}_I(\mathbf{y}_T)$ and $\mathcal{S}_I(\mathbf{y}_T) \subset \mathcal{S}_I(\mathbf{x}_T)$. Moreover, observe that the proof of Theorem 9 proceeds in two steps: in the first step it is shown that any two GNEs admit the same total investment in the CPR, and in the second step it is shown that, given an optimal total investment, every player has a unique optimal investment in the CPR. In the following section we are going to improve upon the aforementioned observations. A bit more concretely, we are going to prove that the set consisting of all GNEs of a Fragile multi-CPR Game is “small" via showing that the set consisting of all total investments at the GNEs is “small".

7 Proof of Theorem 4

Throughout this section, we denote by G a Fragile multi-CPR Game satisfying Assumption 2. Moreover, given a finite set, F, we denote by $|F|$ its cardinality. Now consider the set

$$\mathcal{N}(G) := \{\mathbf{x} \in \mathcal{C}_n : \mathbf{x} \text{ is a GNE of } G\}$$

and, given $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{N}(G)$, let

$$\mathcal{T}_I(\mathbf{x}) = \{i \in [n] : \mathbf{x}_i \text{ is of Type I}\}$$

and

$$\mathcal{T}_{II}(\mathbf{x}) = \{i \in [n] : \mathbf{x}_i \text{ is of Type II}\} .$$

Recall the definition of active CPRs corresponding to $\mathbf{x}_{-i}$, which is denoted $A(\mathbf{x}_{-i})$ and is defined in (9), as well as the definition of effective CPRs corresponding to $\mathbf{x}_{-i}$, which is denoted $J_{\mathbf{x}_{-i}}$ and is defined in (18).

Lemma 5 *Let* $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{N}(G)$ *and suppose that* $i \in \mathcal{T}_I(\mathbf{x})$*, for some* $i \in [n]$*. Then it holds* $J_{\mathbf{x}_{-i}} = A(\mathbf{x}_{-i})$*.*

Proof Recall from Theorem 8 that $J_{\mathbf{x}_{-i}}$ is such that $x_{ij} > 0$ if and only if $j \in J_{\mathbf{x}_{-i}}$. Suppose, towards arriving at a contradiction, that there exists $j \in A(\mathbf{x}_{-i}) \setminus J_{\mathbf{x}_{-i}}$. Since $i \in \mathcal{T}_I(\mathbf{x})$, it follows that $\sum_{j\in J_{\mathbf{x}_{-i}}} x_{ij} < 1$ and thus player i can increase her utility by investing a suitably small amount $\varepsilon > 0$ in the j-th CPR. This contradicts the fact that $\mathbf{x}$ is a GNE, and the lemma follows. □

Lemma 6 *Let* $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ *and* $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_n)$ *be two elements from* $\mathcal{N}(G)$ *such that* $\mathbf{x}_T^{(j)} \le \mathbf{y}_T^{(j)}$*, for all* $j \in [m]$*. Then the following hold true:*

1. *If* $i \in \mathcal{T}_I(\mathbf{x})$*, then* $J_{\mathbf{y}_{-i}} \subset J_{\mathbf{x}_{-i}}$*.*
2. *It holds* $\mathcal{T}_I(\mathbf{x}) \subset \mathcal{T}_I(\mathbf{y})$*.*

Proof Fix $i \in [n]$ such that $i \in \mathcal{T}_I(\mathbf{x})$ and notice that Lemma 5 implies that $\mathbf{x}_T^{(j)} \ge \omega_{ij}$, for all $j \in [m] \setminus J_{\mathbf{x}_{-i}}$. Since $\mathbf{x}_T^{(j)} \le \mathbf{y}_T^{(j)}$, for all $j \in [m]$, it holds $\mathbf{y}_T^{(j)} \ge \omega_{ij}$, for all $j \in [m] \setminus J_{\mathbf{y}_{-i}}$, and we conclude that $J_{\mathbf{y}_{-i}} \subset J_{\mathbf{x}_{-i}}$. The first statement follows.

We proceed with the second statement. Let $i \in [n]$ be such that $\mathbf{x}_i$ is of Type I. We have to show that $\mathbf{y}_i$ is also of Type I. Suppose that this is not true; hence $\mathbf{y}_i$ is of Type II, and thus it holds $\sum_{j\in J_{\mathbf{y}_{-i}}} y_{ij} = 1$. Since $\mathbf{y}_i$ is of Type II, it follows from (27) that $\mathcal{H}_{ij}(\mathbf{y}_T^{(j)}; \kappa_0) = y_{ij}$, for all $j \in J_{\mathbf{y}_{-i}}$ and some $\kappa_0 \ge 0$. Since $\mathbf{x}_i$ is of Type I and $\mathcal{H}_{ij}$ is decreasing, we may apply (26) and conclude

$$x_{ij} = \mathcal{G}_{ij}(\mathbf{x}_T^{(j)}) \ge \mathcal{H}_{ij}(\mathbf{x}_T^{(j)}; \kappa_0) \ge \mathcal{H}_{ij}(\mathbf{y}_T^{(j)}; \kappa_0) = y_{ij}, \quad \text{for all } \; j \in J_{\mathbf{y}_{-i}} .$$

Hence $1 > \sum_{j\in J_{\mathbf{y}_{-i}}} x_{ij} \ge \sum_{j\in J_{\mathbf{y}_{-i}}} y_{ij} = 1$, a contradiction. The result follows. □

Lemma 7 *Assume that* $m \le n$*. Then it holds* $\mathcal{T}_I(\mathbf{x}) \ne \emptyset$*, for every* $\mathbf{x} \in \mathcal{N}(G)$*.*

Proof Suppose that the conclusion is not true; hence there exists $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{N}(G)$ such that $\mathcal{T}_{II}(\mathbf{x}) = [n]$, which in turn implies that $\sum_{j\in[m]} \mathbf{x}_T^{(j)} = n \ge m$. Hence there exists $k \in [m]$ such that $\mathbf{x}_T^{(k)} \ge 1$. We now claim that $\mathbf{x}_T^{k|i} \ge \omega_{ik}$, for all $i \in \mathcal{S}_{II}(\mathbf{x}_T^{(k)})$, where $\mathcal{S}_{II}(\cdot)$ is defined in (30) and ω_{ik} is given by Lemma 1. To prove the claim, notice that if there exists $i \in \mathcal{S}_{II}(\mathbf{x}_T^{(k)})$ such that $\mathbf{x}_T^{k|i} < \omega_{ik}$ then, since x_{ik} is a best response of player i in the k-th CPR, by Remark 1, it would assume a value for which $x_{ik} + \mathbf{x}_T^{k|i} < \omega_{ij}$, which contradicts the fact that $\mathbf{x}_T^{(k)} \ge 1$. The claim follows.

However, since x_{ik} is a best response and $\mathbf{x}_T^{k|i} \ge \omega_{ik}$, for all $i \in \mathcal{S}_{II}(\mathbf{x}_T^{(k)})$, it follows that $x_{ik} = 0$, for all $i \in \mathcal{S}_{II}(\mathbf{x}_T^{(k)})$. This contradicts the fact that $\mathbf{x}_T^{(k)} \ge 1$, and the result follows. □

Lemma 8 *Assume that $m \leq n$. Then there do not exist distinct elements $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_n)$ in $\mathcal{N}(G)$ for which it holds $\mathbf{x}_T^{(j)} \leq \mathbf{y}_T^{(j)}$, for all $j \in [m]$, and $\sum_{j\in[m]} \mathbf{x}_T^{(j)} < \sum_{j\in[m]} \mathbf{y}_T^{(j)}$.*

Proof The proof is deferred to "Appendix D". □

Finally, the proof of Theorem 4 requires the following measure-theoretic results. Here and later, given a positive integer $k \geq 1$, $\mathcal{L}^k$ denotes k-dimensional Lebesgue measure. Moreover, given a function $f : \mathbb{R}^k \to \mathbb{R}^m$ and a set $B \subset \mathbb{R}^m$, we denote $f^{-1}(B) := \{\mathbf{x} \in \mathbb{R}^k : f(\mathbf{x}) \in B\}$ the preimage of B under f.

Lemma 9 *Let $f : \mathbb{R}^d \to \mathbb{R}^m$ be a continuously differentiable function for which it holds $\mathcal{L}^d(\{\mathbf{x} \in \mathbb{R}^d : \nabla f(\mathbf{x}) = 0\}) = 0$. Then we have $\mathcal{L}^d(f^{-1}(A)) = 0$, for every $A \subset \mathbb{R}^m$ for which $\mathcal{L}^m(A) = 0$.*

Proof See Ponomarev (1987, Theorem 1). □

Let $m \geq 1$ be an integer. A set $A \subset [0,1]^m$ is called an *antichain* if it does not contain two distinct elements $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ such that $x_j \leq y_j$, for all $j \in [m]$.

Lemma 10 *Let $A \subset [0,1]^m$ be an antichain. Then $\mathcal{L}^m(A) = 0$.*

Proof The result is an immediate consequence of Lebesgue's density theorem. Alternatively, it follows from the main result in Engel (1986), and from Engel et al. (2020, Theorem 1.3). □

Now given $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{C}_n$, let $\mathbf{v}_\mathbf{x}$ denote the vector

$$\mathbf{v}_\mathbf{x} := (\mathbf{x}_T^{(1)}, \ldots, \mathbf{x}_T^{(m)}) \in [0,1]^m, \tag{32}$$

where $\mathbf{x}_T^{(j)}$, $j \in [m]$, is defined in (4). Finally, given $N \subset \mathcal{C}_n$, define the set

$$W_N := \bigcup_{\mathbf{x}\in N} \mathbf{v}_\mathbf{x} \,. \tag{33}$$

The proof of Theorem 4 is almost complete.

Proof (Proof of Theorem 4) To simplify notation, let us set $N := \mathcal{N}(G)$. We have to show that $\mathcal{L}^{nm}(N) = 0$.

Now let f denote the map $f : \mathcal{C}_n \to [0,1]^m$ given by $f(\mathbf{x}) = \mathbf{v}_\mathbf{x}$, where $\mathbf{v}_\mathbf{x}$ is defined in (32). It is straightforward to verify that $\{\mathbf{x} \in \mathcal{C}_n : \nabla f(\mathbf{x}) = 0\} = \emptyset$.

Now consider the set W_N, defined in (33), and notice that Lemma 8 implies that W_N is an antichain; hence it follows from Lemma 10 that $\mathcal{L}^m(W_N) = 0$. Therefore, Lemma 9 yields

$$\mathcal{L}^{nm}(N) = \mathcal{L}^{nm}(f^{-1}(W_N)) = 0\,,$$

as desired. □

8 A restricted version of the game

Let G denote a Fragile multi-CPR Game satisfying Assumption 2. In this section we show that G admits finitely many GNEs, subject to the constraint that the total investment in each CPR is fixed. We then use this result, in the next section, in order to formulate a conjecture which is equivalent to Conjecture 1. Before being more precise, we need some extra piece of notation.

Given a set $F \subset [m]$ and real numbers $\{r_j\}_{j\in F} \subset [0, 1]$, indexed by F, we denote by $W(\{r_j\}_{j\in F})$ the set

$$W(\{r_j\}_{j\in F}) := \{\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{C}_n : \mathbf{x}_T^{(j)} = r_j, \text{ for } j \in F\},$$

where $\mathbf{x}_T^{(j)}$ is defined in (4). In other words, $W(\{r_j\}_{j\in F})$ consists of those strategy profiles for which the total investment in the CPRs corresponding to elements in F is fixed, and equal to the given numbers $\{r_j\}_{j\in F}$.

In this section we prove the following.

Theorem 10 *Fix real numbers $r_1, \ldots, r_m \in [0, 1]$. Then the set $W := W(r_1, \ldots, r_m)$ contains at most $2^{n\cdot(m+1)}$ GNEs of G.*

The proof requires a couple of observations which we collect in the following lemmata.

Lemma 11 *Suppose that $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_n)$ are two GNEs of G such that $\mathbf{x}, \mathbf{y} \in W := W(r_j)$ and $0 < x_{ij} < y_{ij}$, for some $i \in [n]$, $j \in [m]$ and $r_j \in [0, 1]$. Then either $\mathbf{x}_i$ is of Type II or $\mathbf{y}_i$ is of Type II.*

Proof Suppose, towards arriving at a contradiction, that the conclusion is not true. Then both $\mathbf{x}_i$ and $\mathbf{y}_i$ are of Type I, and thus (26) implies that $\mathcal{G}_{ij}(r) = x_{ij}$ and $\mathcal{G}_{ij}(r) = y_{ij}$. Hence it holds $\mathcal{G}_{ij}(r_j) = x_{ij} < y_{ij} = \mathcal{G}_{ij}(r_j)$, a contradiction. The result follows. □

Lemma 12 *Suppose that $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_n)$ are two GNEs of G for which it holds $\mathbf{x}, \mathbf{y} \in W := W(r_l, r_\ell)$ and $0 < x_{ij} < y_{ij}$ and $x_{i\ell} > y_{i\ell} > 0$, for some $i \in [n]$ and $\{j, \ell\} \subset [m]$. Then either $\mathbf{x}_i$ is of Type I or $\mathbf{y}_i$ is of Type I.*

Proof Suppose, towards arriving at a contradiction, that both $\mathbf{x}_{-i}$ and $\mathbf{y}_{-i}$ are of Type II. Recall the definition of $\psi_{ij}(\cdot\,;\,\cdot)$, given in (14), and notice that, since both $\mathbf{x}_i, \mathbf{y}_i$ are of Type II, Theorem 8 implies the existence of $\kappa_x, \kappa_y \geq 0$ such that

$$\kappa_x = x_{ij}^{a_i-1} \cdot \psi_{ij}(x_{ij}; r_j - x_{ij}) = x_{i\ell}^{a_i-1} \cdot \psi_{i\ell}(x_{i\ell}; r_\ell - x_{i\ell})$$

and

$$\kappa_y = y_{ij}^{a_i-1} \cdot \psi_{ij}(y_{ij}; r_j - y_{ij}) = y_{i\ell}^{a_i-1} \cdot \psi_{i\ell}(y_{i\ell}; r_\ell - y_{i\ell})\,.$$

Now notice that, for all $k \in [m]$, the function $\Psi_k(x) := x^{a-1} \cdot \psi_{ik}(x; r - x)$ is decreasing in x, for fixed $r > 0$ and $a \in (0, 1]$. Hence, $x_{ij} < y_{ij}$ implies that $\kappa_x > \kappa_y$, and $x_{i\ell} > y_{i\ell}$ implies that $\kappa_x < \kappa_y$, a contradiction. The result follows. □

We may now proceed with the proof of the main result of this section.

Proof (Proof of Theorem 10) For every $i \in [n]$, define the set

$$N_i := \{\mathbf{x}_i \in C_m : (\mathbf{x}_i, \mathbf{x}_{-i}) \in W, \text{ for some } \mathbf{x}_{-i} \in \mathcal{C}_{-i}\}.$$

We first show that the cardinality of N_i, denoted $|N_i|$, is at most 2^{m+1}.

Let $\mathbf{x}_i \in N_i$, and recall from Theorem 8, and (18), that there exists $J \subset [m]$ such that $x_{ij} > 0$, when $j \in J$, and $x_{ij} = 0$ when $j \in [m] \setminus J$. In other words, to every $\mathbf{x}_i \in N_i$ there corresponds a set $J \subset [m]$ such that $x_{ij} > 0$ if and only if $j \in J$. Now, given $J \subset [m]$, let

$$N_J := \{\mathbf{x}_i \in N_i : x_{ij} > 0 \text{ if and only if } j \in J\}.$$

Assume first that $|J| \geq 2$. In this case we claim that $|N_J| \leq 2$. Indeed, if $|N_J| \geq 3$, then there are two elements, say $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in N_J$, which are either both of Type I, or both of Type II. If both $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are of Type I, then there exists $j \in J$ such that, without loss of generality, it holds $x_{ij}^{(1)} < x_{ij}^{(2)}$; which contradicts Lemma 11. If both $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are of Type II, then there exist $j, \ell \in J$ such that $x_{ij} < y_{ij}$ and $x_{i\ell} > y_{i\ell}$; which contradicts Lemma 12. The claim follows.

If $|J| = 1$, say $J = \{j\}$, we claim that $|N_J| \leq 1$. Indeed, suppose that $|N_J| \geq 2$ holds true and notice that every element of N_J is of Type I. However, the assumption that $|N_J| \geq 2$ implies that there exist $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in N_J$ such that $0 < x_{ij}^{(1)} < y_{ij}^{(1)}$; which contradicts Lemma 11. The second claim follows.

Since there are 2^m subsets $J \subset [m]$, and for each J it holds $|N_J| \leq 2$, it follows that there are at most 2^{m+1} elements in N_i. Since there are n players in the game, the result follows. □

9 Concluding remarks

9.1 Future work

Throughout this subsection, let G denote a Fragile multi-CPR Game satisfying Assumption 2, and let $\mathcal{N}(G)$ be the set consisting of all GNEs of G. So far we have proven that the $(n \cdot m)$-dimensional Lebesgue measure of $\mathcal{N}(G)$ equals zero, but there are several problems and questions that remain open. First and foremost, we believe that the following holds true.

Conjecture 2 Let $N := \mathcal{N}(G)$. Then the antichain W_N, defined in (33), is finite.

Notice that if Conjecture 2 holds true then, in view of Theorem 10, Conjecture 1 holds true as well. Since the converse is clearly true, it follows that Conjecture 1 and Conjecture 2 are equivalent. The exact number of GNEs in a Fragile multi-CPR Game appears to depend on the relation between the number of players, n, and the number of CPRs, m. When $n \geq m$ we conjecture that that for every GNE the players choose best responses of Type I and therefore, provided this is indeed the case, Theorem 9 would imply that the game admits a unique GNE.

Conjecture 3 If $n \geq m$, then $|\mathcal{N}(G)| = 1$.

Another line of research is to investigate the *best response dynamics* of a Fragile multi-CPR Game, which may be seen as a behavioral rule along which players fix an initial investment in the CPRs and proceed with updating their investment, over rounds, in such a way that in the t-th round player $i \in [n]$ invests $\mathbf{b}_i^{(t)} := B_i(\mathbf{x}_{-i}^{(t)})$, where $B_i(\cdot)$ is defined in (13) and $\mathbf{x}_{-i}^{(t)} \in \mathcal{C}_{-i}$ is the strategy profile of all players except player i in the t-th round. A natural question to ask is whether the best response dynamics converge, i.e., whether there exists a round t_0 such that $\mathbf{b}_i^{(t)} = \mathbf{b}_i^{(t_0)}$, for all $t \geq t_0$ and all $i \in [n]$.

Conjecture 4 The best response dynamics of G converge.

When $m = 1$, it is shown in Hota et al. (2016) that the best response dynamics of a Fragile CPR Game converge to its Nash Equilibrium. This is obtained as a consequence of the fact that the best response correspondence is single-valued and decreasing in the total investment in the CPR [see the remarks following Hota et al. (2016, Proposition 7)]. Moreover, it is not difficult to verify that the Nash equilibrium of a Fragile CPR Game is also a Generalized Nash equilibrium. Hence, the best response dynamics of a Fragile CPR Game converge to the Generalized Nash equilibrium. When $m \geq 2$, the best response correspondence need no longer be decreasing in each CPR. It is decreasing for those players whose best response is of Type I, as can be easily seen using the fact that the auxiliary function $\mathcal{G}_{ij}$ is decreasing. This monotonicity may no longer hold true when a player moves from a best response of Type II to a best response of Type I, or from a best response of Type II to a best response of the same type. Furthermore, Theorem 8 does not guarantee that the set of effective CPR, defined in (18), is unique. Hence, the best response correspondence may not be single-valued. So far our theoretical analysis does not provide sufficient evidence for the holistic validity of Conjecture 4. However, our numerical experiments suggest that Conjecture 4 holds true. Finally, let us remark that a natural question is to investigate what happens when the assumption on independence between different CPRs is dropped. We expect that we will be able to report on those matters in the future.

9.2 An example

In this subsection we discuss an example that illustrates the, rather intuitive, fact that multi-CPR games, when compared to single-CPR games, do not decrease the utility of the players and do not increase the probability that a particular CPR fails.

Let C denote a CPR having rate of return $\mathcal{R}(\mathbf{x}_T) = 2 + \mathbf{x}_T$ and probability of failure $p(\mathbf{x}_T) = \mathbf{x}_T$, and let $G^{(1)}$ denote a Fragile CPR Game played on C with $n = 2$ players. Assume further that $a_i = k_i = 1$, for $i \in \{1, 2\}$. It is straightforward to verify that the function $\mathcal{F}$, defined in (2), satisfies Assumption 1 and is given by

$\mathcal{F}(\mathbf{x}_T) = 1 - \mathbf{x}_T - \mathbf{x}_T^2$; hence the coefficient ω, provided by Lemma 1, is equal to $\frac{\sqrt{5}-1}{2} \approx 0.61803$ for both players. Let (x_0, y_0) be the Nash equilibrium of $G^{(1)}$. Since $x_0, y_0 < 1$, it follows that both x_0 and y_0 are of Type I. Furthermore, since the utilities of both players are the same and the NE is unique, it follows that $x_0 = y_0$. Now it is not difficult to verify that the first order condition of Theorem 8 states that x_0 is the solution of the equation $8x^2 + 3x - 1 = 0$. Thus $x_0 = \frac{\sqrt{41}-3}{16} \approx 0.21269$, and we conclude that $(\frac{\sqrt{41}-3}{16}, \frac{\sqrt{41}-3}{16})$ is the Nash equilibrium of $G^{(1)}$, and that the utility of each player is equal to $x_0 \cdot \mathcal{F}(2x_0) \approx 0.08372$. The probability that the CPR fails is equal to $p(2x_0) \approx 0.42539$.

Now, let $G^{(2)}$ denote a Fragile multi-CPR with $n = 2$ players and $m = 2$ independent CPRs that are identical to C. Since $2x_0 < 1$, it is easy to verify that a GNE of $G^{(2)}$ is the tuple $(\mathbf{x}_1, \mathbf{x}_2)$, where $\mathbf{x}_i = (x_0, x_0)$, for $i \in \{1, 2\}$. Notice that the utilities of the players, corresponding to this particular GNE, are doubled when compared to the utilities corresponding to the NE of $G^{(1)}$, and the probability that a particular CPR fails is the same as in the case of $G^{(1)}$. Similar conclusions can be drawn for the case $m \in \{3, 4\}$: each player invests x_0 is each CPR, and the GNEs appear to be unique. Let us now proceed with discussing cases having more CPRs.

Let us first discuss the case $m = 6$. Let $G^{(6)}$ be a Fragile multi-CPR Game with $n = 2$ players and *six* independent CPRs each of which is identical to C. Assume further that $a_i = k_i = 1$, for $i \in \{1, 2\}$. Let $\mathbf{x}_i = (x_{i1}, \ldots, x_{i6})$, $i \in \{1, 2\}$, be two vectors such that $(\mathbf{x}_1, \mathbf{x}_2)$ is a GNE of $G^{(6)}$. Then we know from Theorem 8 that the vectors $\mathbf{x}_1, \mathbf{x}_2$ are either of Type I or of Type II. Let us assume that both $\mathbf{x}_1$ and $\mathbf{x}_2$ are of Type II. Then Theorem 8 implies that there exists $J_1 \subset \{1, \ldots, 6\}$ and $\kappa_1 \geq 0$ such that $\sum_{j \in J_1} x_{1j} = 1$ and

$$x_{1j}(-1 - 2\mathbf{x}_T^{(j)}) + 1 - \mathbf{x}_T^{(j)} - (\mathbf{x}_T^{(j)})^2 = \kappa_1, \quad \text{for all } \ j \in J_1 \,. \tag{34}$$

Similarly, there exists $J_2 \subset \{1, \ldots, 6\}$ and $\kappa_2 \geq 0$ such that $\sum_{j \in J_2} x_{2j} = 1$ and

$$x_{2j}(-1 - 2\mathbf{x}_T^{(j)}) + 1 - \mathbf{x}_T^{(j)} - (\mathbf{x}_T^{(j)})^2 = \kappa_2, \quad \text{for all } \ j \in J_2 \,. \tag{35}$$

Now consider the points $\mathbf{y}_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0)$ and $\mathbf{y}_2 = (0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. It is easy to see that the coordinates of $\mathbf{y}_1, \mathbf{y}_2$ satisfy (34) and (35) with $\kappa_1 = \kappa_2 = 0$ and $J_1 = \{1, 2, 3\}$, $J_2 = \{4, 5, 6\}$, and that the tuple $(\mathbf{y}_1, \mathbf{y}_2)$ is a Nash equilibrium of $G^{(6)}$. It is also clear from this example that a GNE of a multi-CPR Game may not be unique: the tuple $(\mathbf{y}_2, \mathbf{y}_1)$ is also a GNE of $G^{(6)}$, and numerical experiments suggest that $G^{(6)}$ admits GNEs for which one player chooses a vector of Type I and the other player chooses a vector of Type II. Moreover, the system of equations given in (34) and (35) may have infinitely many solutions and, roughly speaking, Conjecture 1 states that at most finitely many of those solutions constitute a GNE of $G^{(6)}$. Notice that the utility of each player at the GNE $(\mathbf{x}_1, \mathbf{x}_2)$ is equal to $\mathcal{F}(1/3) = \frac{5}{9} \approx 0.55555$ and the probability that a particular CPR fails is equal to $p(1/3) = 1/3 \approx 0.33333$. Finally, observe that the above-mentioned GNEs of $G^{(6)}$ satisfy $J_1 \cap J_2 = \emptyset$

If $m = 5$, then the situation is similar to the case $m = 6$: the corresponding "first order conditions" are the same as (34) and (35), with the only difference that they hold true for sets $J_1, J_2 \subset \{1, \dots, 5\}$. However, it is not difficult to see that in this case it holds $J_1 \cap J_2 \neq \emptyset$, and thus the players are forced to share a CPR. Now observe that the coordinates of the vectors $\mathbf{y}_1 = \mathbf{y}_2 = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ satisfy the first order conditions of $G^{(5)}$, with $\kappa_1 = \kappa_2 = \frac{2}{5}$ and $J_1 = J_2 = \{1, \dots, 5\}$. We conjecture that the tuple $(\mathbf{y}_1, \mathbf{y}_2)$ is a GNE of $G^{(5)}$, but we are unable to provide a prove. In order to verify the aforementioned conjecture, we have to show that, letting $\gamma := \omega - \frac{1}{5} \approx 0.41803$, for all $x_1, \dots, x_5 \in (0, \gamma)$ for which $\sum_{j=1}^{5} x_j = 1$, it holds

$$\phi(x_1, \dots, x_5) := \sum_{j=1}^{5} x_j \cdot \mathcal{F}(x_j + \frac{1}{5}) \leq \mathcal{F}(\frac{2}{5}) \,. \tag{36}$$

There should be a nice proof of (36) which uses some Jensen-type argument, but we are unable to find. Some evidence that the conjecture holds can be obtained using standard methods from the theory of Lagrange multipliers: if we consider the more general problem of maximising $\phi(x_1, \dots, x_5)$ subject to the constraint $\sum_{i=1}^{5} x_i = 1$, then one easily sees that the borderer Hessian of the corresponding Lagrange function evaluated at $(1/5, \dots, 1/5)$ alternates sign, which in turn implies that the point $(1/5, \dots, 1/5)$ is a local maximum. Finally, it is easy to verify that the point $(\mathbf{y}_1, \mathbf{y}_2)$, provided it is a GNE, achieves higher utilities of the players (when compared to the utilities of the GNE of $G^{(1)}$) as well as lower probabilities for a particular CPR to fail. Numerical experiments suggest that the GNE of $G^{(5)}$ is not unique.

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A Appendix

B Proof of Lemma 3

Suppose that there exist two distinct vectors, say $(c, x_1, \dots, x_s)$ and $(d, y_1, \dots, y_s)$. If $c = d$, then there exists $j \in [s]$ such that $x_j \neq y_j$ and

$$f_j(x_j) = c = d = f_j(y_j) \,,$$

contrariwise to the assumption that the function $f_i(\cdot)$ is strictly decreasing. Hence $c \neq d$.

Since $f_j(\cdot)$, $j \in [s]$, is strictly decreasing, it is injective and therefore it follows that it is invertible. Let us denote its inverse by $f_j^{-1}(\cdot)$. We then have

$$x_j = f_j^{-1}(c) \text{ and } y_j = f_j^{-1}(d), \text{ for all } j \in [s],$$

which in turn implies that $x_j \neq y_j$, for all $j \in [s]$. Assume, without loss of generality, that $c < d$. The assumption that f_j is strictly decreasing then implies $x_j > y_j$, for all $j \in [s]$, and therefore $1 = \sum_{j\in[s]} x_j > \sum_{j\in[s]} y_j = 1$, a contradiction. The result follows.

C Proof of Lemma 4

To simplify notation, let $\mathcal{F} := \mathcal{F}_{ij}(x_{ij} + \mathbf{x}_T^{j|i})$, $\mathcal{F}' := \frac{\partial}{\partial x_{ij}}\mathcal{F}$ and $\mathcal{F}'' := \frac{\partial^2}{\partial x_{ij}^2}\mathcal{F}$. For $x_{ij} \in (0, \omega_{ij} - \mathbf{x}_T^{(j)})$, we compute

$$\begin{aligned}\frac{\partial}{\partial x_{ij}}\mathcal{H}_{ij}(x_{ij} + \mathbf{x}_T^{j|i}; \kappa_0) &= \frac{-a_i\mathcal{F}' \cdot (\frac{-\kappa_0}{x_{ij}^{a_i}} + \mathcal{F}') + a_i\mathcal{F} \cdot (-a_i\frac{-\kappa_0}{x_{ij}^{a_i+1}} + \mathcal{F}'')}{(\frac{-\kappa_0}{x_{ij}^{a_i}} + \mathcal{F}')^2} \\ &= a_i \cdot \frac{\kappa_0 x_{ij}^{a_i-1}\left(x_{ij}\mathcal{F}' + a_i\mathcal{F}\right) - x_{ij}^{2a_i}(\mathcal{F}')^2 + x_{ij}^{2a_i}\mathcal{F}\cdot\mathcal{F}''}{(-\kappa_0 + x_{ij}^{a_i}\mathcal{F}')^2} \\ &< a_i \cdot \frac{\kappa_0 x_{ij}^{a_i-1}\left(x_{ij}\mathcal{F}' + a_i\mathcal{F}\right) - x_{ij}^{2a_i}(\mathcal{F}')^2}{(-\kappa_0 + x_{ij}^{a_i}\mathcal{F}')^2},\end{aligned}$$

where the last estimate follows from the fact that, by Assumption 2, it holds $\mathcal{F}'' < 0$. If $\kappa_0 = 0$, then it readily follows that $\frac{\partial}{\partial x_{ij}}\mathcal{H}_{ij}(x_{ij} + \mathbf{x}_T^{j|i}; \kappa_0) < 0$ and therefore $\mathcal{H}_{ij}$ is strictly decreasing; thus $\mathcal{G}_{ij}$ is strictly decreasing as well. So we may assume that $\kappa_0 > 0$. If $x_{ij}\mathcal{F}' + a_i\mathcal{F} < 0$, then it also follows that $\mathcal{H}_{ij}$ is strictly decreasing; thus we may also assume that $A := x_{ij}\mathcal{F}' + a_i\mathcal{F} \geq 0$. Now notice that $\frac{\partial A}{\partial x_{ij}} = \mathcal{F}' + x_{ij}\mathcal{F}'' + a_i\mathcal{F}' < 0$, and define the function

$$H(x_{ij}) := \kappa_0 x_{ij}^{a_i-1} \cdot A - x_{ij}^{2a_i}(\mathcal{F}')^2 ;$$

hence it holds $\frac{\partial}{\partial x_{ij}}\mathcal{H}_{ij}(x_{ij} + \mathbf{x}_T^{j|i}; \kappa_0) < a_i \cdot \frac{H(x_{ij})}{(-\kappa_0 + x_{ij}^{a_i}\mathcal{F}')^2}$. Moreover, it holds

$$\frac{\partial}{\partial x_{ij}}H(x_{ij}) = (a_i - 1)\kappa_0 x_{ij}^{a_i-2} \cdot A + \kappa_0 x_{ij}^{a_i-1} \cdot \frac{\partial A}{\partial x_{ij}} - 2a_i x_{ij}^{2a_i-1}(\mathcal{F}')^2 - 2x_{ij}^{2a_i}\mathcal{F}'\mathcal{F}''.$$

Since $a_i \leq 1$, $A \geq 0$ and $\mathcal{F}', \mathcal{F}'', \frac{\partial A}{\partial x_{ij}} < 0$, it readily follows that all addends in the previous equation are negative, and therefore $\frac{\partial}{\partial x_{ij}}H(x_{ij}) < 0$. In other words, $H(\cdot)$ is strictly decreasing in $[0, \omega_{ij} - \mathbf{x}_T^{j|i}]$ and, since $H(0) = 0$, $H(\omega_{ij} - \mathbf{x}_T^{j|i}) < 0$, we conclude that $H(x_{ij}) \leq 0$, for all $x_{ij} \in [0, \omega_{ij} - \mathbf{x}_T^{j|i}]$. This implies that $\frac{\partial}{\partial x_{ij}}\mathcal{H}_{ij}(x_{ij} + \mathbf{x}_T^{j|i}; \kappa_0) < 0$ for $x_{ij} \in [0, \omega_{ij} - \mathbf{x}_T^{j|i}]$. Since $\frac{\partial}{\partial \mathbf{x}_T^{(j)}}\mathcal{H}_{ij}(\mathbf{x}_T^{(j)}; \kappa_0) =$

$\frac{\partial}{\partial x_{ij}}\mathcal{H}_{ij}(x_{ij}+\mathbf{x}_T^{j|i};\kappa_0)$, and similarly for $\mathcal{G}_{ij}$, we conclude that both $\mathcal{G}_{ij}$ and $\mathcal{H}_{ij}$ are strictly decreasing in the interval $[0,\omega_{ij}]$, as desired.

D Proof of Lemma 8

Suppose that such GNEs do exist. Since $m \le n$, it follows from Lemma 7 that $\mathcal{T}_I(\mathbf{x}) \neq \emptyset$, for every $\mathbf{x} \in \mathcal{N}(G)$.

Notice that Lemma 6 implies that $\mathcal{T}_I(\mathbf{x}) \subset \mathcal{T}_I(\mathbf{y})$ and $J_{\mathbf{y}_{-i}} \subset J_{\mathbf{x}_{-i}}$, and (26) implies that $\mathcal{G}_{ij}(\mathbf{x}_T^{(j)}) = x_{ij}$, for all $i \in \mathcal{T}_I(\mathbf{x})$ and all $j \in J_{\mathbf{x}_{-i}}$. Similarly, it holds $\mathcal{G}_{ij}(\mathbf{y}_T^{(j)}) = y_{ij}$, for all $i \in \mathcal{T}_I(\mathbf{y})$ and all $j \in J_{\mathbf{y}_{-i}}$. Hence we may write

$$\sum_{j\in[m]} \mathbf{x}_T^{(j)} = \sum_{i\in\mathcal{T}_I(\mathbf{x})}\sum_{j\in J_{\mathbf{x}_{-i}}} \mathcal{G}_{ij}(\mathbf{x}_T^{(j)}) + |\mathcal{T}_{II}(\mathbf{x})|$$

as well as

$$\sum_{j\in[m]} \mathbf{y}_T^{(j)} = \sum_{i\in\mathcal{T}_I(\mathbf{y})}\sum_{j\in J_{\mathbf{y}_{-i}}} \mathcal{G}_{ij}(\mathbf{y}_T^{(j)}) + |\mathcal{T}_{II}(\mathbf{y})|\,.$$

Since $\sum_j \mathbf{x}_T^{(j)} < \sum_j \mathbf{y}_T^{(j)}$, $|\mathcal{T}_I(\mathbf{x})| \le |\mathcal{T}_I(\mathbf{y})|$ and $|\mathcal{T}_{II}(\mathbf{x})| \ge |\mathcal{T}_{II}(\mathbf{y})|$ hold true, it follows that

$$\begin{aligned}\sum_{i\in\mathcal{T}_I(\mathbf{x})}\sum_{j\in J_{\mathbf{x}_{-i}}} \mathcal{G}_{ij}(\mathbf{x}_T^{(j)}) + |\mathcal{T}_{II}(\mathbf{x}) \setminus \mathcal{T}_{II}(\mathbf{y})| \;<\; &\sum_{i\in\mathcal{T}_I(\mathbf{x})}\sum_{j\in J_{\mathbf{y}_{-i}}} \mathcal{G}_{ij}(\mathbf{y}_T^{(j)}) \\ &+ \sum_{i\in\mathcal{T}_I(\mathbf{y})\setminus\mathcal{T}_I(\mathbf{x})}\;\sum_{j\in J_{\mathbf{y}_{-i}}} \mathcal{G}_{ij}(\mathbf{y}_T^{(j)})\,. \end{aligned} \tag{37}$$

However, the fact that $\mathcal{G}_{ij}$ is decreasing implies that $\mathcal{G}_{ij}(\mathbf{x}_T^{(j)}) \ge \mathcal{G}_{ij}(\mathbf{y}_T^{(j)})$, for all $i \in \mathcal{T}_I(\mathbf{x})$ and all $j \in J_{\mathbf{y}_{-i}}$; hence it holds

$$\sum_{i\in\mathcal{T}_I(\mathbf{x})}\sum_{j\in J_{\mathbf{x}_{-i}}} \mathcal{G}_{ij}(\mathbf{x}_T^{(j)}) \ge \sum_{i\in\mathcal{T}_I(\mathbf{x})}\sum_{j\in J_{\mathbf{y}_{-i}}} \mathcal{G}_{ij}(\mathbf{y}_T^{(j)})\,. \tag{38}$$

Moreover, since $\sum_{j\in J_{\mathbf{y}_{-i}}} \mathcal{G}_{ij}(\mathbf{y}_T^{(j)}) < 1$, for all $i \in \mathcal{T}_I(\mathbf{y}) \setminus \mathcal{T}_I(\mathbf{x})$, it holds

$$\sum_{i\in\mathcal{T}_I(\mathbf{y})\setminus\mathcal{T}_I(\mathbf{x})}\;\sum_{j\in J_{\mathbf{y}_{-i}}} \mathcal{G}_{ij}(\mathbf{y}_T^{(j)}) < |\mathcal{T}_I(\mathbf{y}) \setminus \mathcal{T}_I(\mathbf{x})| = |\mathcal{T}_{II}(\mathbf{x}) \setminus \mathcal{T}_{II}(\mathbf{y})|\,. \tag{39}$$

Now notice that (38) and (39) contradict (37). The result follows.

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